On the b-coloring of P_4-tidy graphs

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Abstract

A *b-coloring* of a graph is a coloring such that every color class admits a vertex adjacent to at least one vertex receiving each of the colors not assigned to it. The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors. A graph G is *b-continuous* if it admits a b-coloring with t colors, for every $t = \chi(G), \ldots, \chi_b(G)$, and it is *b-monotonic* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1. In this work, we prove that P_4-tidy graphs (a generalization of many classes of graphs with few induced P_4s) are b-continuous and b-monotonic. Furthermore, we describe a polynomial time algorithm to compute the b-chromatic number for this class of graphs.

Key words: b-coloring, b-continuity, b-monotonicity, P_4-tidy graphs

1. Introduction

In this paper we deal with finite undirected graphs, without loops or multiple edges. A *coloring* of a graph G is an assignment of colors (represented by natural numbers) to the vertices of G such that no two adjacent vertices are assigned the same color. The minimum number k such that there exists a coloring of G with k colors is the *chromatic number* of G, denoted by $\chi(G)$.

When we try to color the vertices of a graph using the minimum number of colors, we can start from a given coloring and try to decrease the number of colors by eliminating color classes. One possible such procedure consists in trying to reduce the number of colors by taking a color class such that we can recolor every vertex from that class with a different color that is not used by any of its neighbors, if any such class exists. A vertex v of a colored graph G is *dominant* if it has at least one neighbor of every color, except the one assigned to v. A dominant vertex cannot be recolored with this procedure. A *b-coloring* of a graph is a coloring with dominant vertices in each color class, i.e, a coloring where we cannot apply the strategy above to decrease the number of colors. The *b-chromatic number* of a

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graph G, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors [15]. Thus, $\chi_b(G) \geq \chi(G)$, and every coloring with $\chi(G)$ colors is a b-coloring. A graph G is *b-perfect* if $\chi_b(H) = \chi(H)$ for every induced subgraph H of G [11]. b-perfect graphs were recently characterized by a finite list of forbidden induced subgraphs [14]. Note that b-perfect graphs can be colored with minimum number of colors in polynomial time, by simply applying the decreasing algorithm, starting from an arbitrary coloring.

The behavior of the b-chromatic number can be surprising. For example, the values of k for which a graph admits a b-coloring with k colors do not necessarily form an interval of the set of integers; in fact any finite subset of $\mathbb{N}_{\geq 2}$ can be the set of these values for some graph [9]. A graph G is *b-continuous* if it admits a b-coloring with t colors, for every $t = \chi(G), \ldots, \chi_b(G)$. In [21] (see also [9]) it is proved that chordal graphs and some planar graphs are b-continuous.

Another atypical property is that the b-chromatic number can increase when taking induced subgraphs. A graph G is defined to be *b-monotonic* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1 [3].

Irving and Manlove [15] proved that determining $\chi_b(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. In [24], Kratochvíl, Tuza and Voigt show that determining $\chi_b(G)$ is NP-hard even if G is a connected bipartite graph. More results on algorithmic aspects and bounds for some graph classes can be found in [3, 7, 8, 16, 23].

An induced path on k vertices shall be denoted by P_k. Vertices of degree one (resp. two) in P_k will be called *endpoints* (resp. *midpoints*). An induced subgraph of G isomorphic to P_k is simply said to be a P_k in G. A chordless cycle on k vertices is denoted by C_k.

A *cograph* is a graph that does not contain P_4 as an induced subgraph [5]. Several generalizations of cographs have been defined in the literature, such as P_4-sparse [13], P_4-lite [17], P_4-extendible [19] and P_4-reducible graphs [18]. A graph class generalizing all of them is the class of P_4-tidy graphs [10]. Let G be a graph and A a P_4 in G. A *partner* of A is a vertex v in $G - A$ such that $A \cup \{v\}$ induces at least two P_4s in G. A graph G is *P_4-sparse* if no induced P_4 has a partner and *P_4-tidy* if every induced P_4 has at most one partner. Another generalization of P_4-sparse graphs are $(q, q4)$-graphs. A graph is a *$(q, q4)$-graph* if no set of at most q vertices induces more than $q-4$ distinct P_4's [1]. There is no containment relationship between the classes P_4-tidy and $(q, q4)$-graphs.

In [3], it was proved that P_4-sparse graphs are b-continuous and b-monotonic and a dynamic programming algorithm to compute their b-chromatic number was presented. Recently, some of the results on P_4-sparse graphs were also extended for the class of $(q, q4)$-graphs, with fixed q [4]. In this paper, we extend these results to the class of P_4-tidy graphs.

1.1. Definitions and preliminary results

Let $G = (V, E)$ be a graph. We will denote by $V(G)$ the vertex set V, by $E(G)$ the edge set E, and by $\overline{G}$ the complement graph of G. Given a subset of vertices $X \subset V$, we will denote by $G[X]$ the subgraph of G induced by X. The complete graph on n vertices will be denoted by K_n and the stable set of n vertices by S_n. Two vertices will be said to be *true twins* if they are adjacent and have the same neighborhood, and *false twins* if they are non-adjacent but have the same neighbors. A vertex is *simplicial* if its neighbors induce a complete subgraph. A vertex v *controls* a vertex w if v and w are non-adjacent and all the neighbors of w are neighbors of v.

Lemma 1. [12] *Let G be a graph and φ a coloring of G. If v and w are false twins in G, then either none of them is dominant, or $\varphi(v) = \varphi(w)$.*

This can be extended straightforward to the following one.

Lemma 2. *Let G be a graph and φ a coloring of G. If v controls w, then if w is dominant, so is v and $\varphi(v) = \varphi(w)$.*

Lemma 3. [12] *Let G be a graph and φ a coloring of G with more than $\chi(G)$ colors. Then no simplicial vertex of G is dominant.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The union is clearly an associative operation and, for each nonnegative integer t, we will denote by tG the union of t disjoint copies of G. The *join* of G_1 and G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$. That is, the vertex set of $G_1 \vee G_2$ is $V_1 \cup V_2$ and its edge set is $E_1 \cup E_2$ plus all the possible edges with an endpoint in V_1 and the other one in V_2.

Cographs can be built from isolated vertices by using these two operations.

Theorem 1. [5] *Every non-trivial cograph is either union or join of two smaller cographs.*

Thus, the chromatic number of a cograph can be recursively calculated due to the following result.

Theorem 2. [6] *If G is the trivial graph, then $\chi(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*
i. $\chi(G_1 \cup G_2) = max\{\chi(G_1), \chi(G_2)\}$
ii. $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.

A similar result holds for the b-chromatic number, but the relation between the b-chromatic number of two graphs and the b-chromatic number of their union is weaker.

Theorem 3. [22] *If G is the trivial graph, then $\chi_b(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*
i. $\chi_b(G_1 \cup G_2) \geq max\{\chi_b(G_1), \chi_b(G_2)\}$
ii. $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2)$.

P_4-tidy graphs have also a useful decomposition theorem. We will use it extensively in this work to inductively prove our results. A brief description of the theorem follows.

Let $G = (V, E)$ be a graph. Let $F = \{e \in E | e \text{ belongs to an induced } P_4 \text{ of } G\}$. Let $G_p = (V, F)$. A connected component of G_p having exactly one vertex is called a *weak vertex*. Any connected component of G_p distinct from a weak vertex is called a *p-component* of G. A graph G is *p-connected* if it has only one p-component and no weak vertices [2].

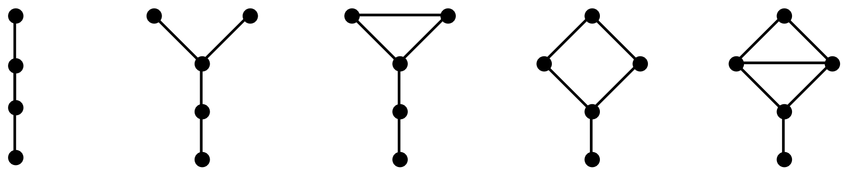

Figure 1: Possible quasi-starfishes of size two. From left to right: P_4, fork, $\overline{P}$, P and kite.

A p-connected graph $G = (V, E)$ is *p-separable* if V can be partitioned into two sets (C, S) such that each P_4 that contains vertices from C and from S has its midpoints in C and its endpoints in S. We will call it a *p-partition.* If such a partition there exists, then it is unique [20].

An *urchin* (resp. *starfish*) of size k, $k \geq 2$, is a p-separable graph with p-partition (C, S), where $C = \{c_1, \ldots, c_k\}$ is a clique; $S = \{s_1, \ldots, s_k\}$ is a stable set; s_i is adjacent to c_i if and only if $i = j$ (resp. $i \neq j$).

A *quasi-urchin* (resp. *quasi-starfish*) of size k is a graph obtained from an urchin (resp. starfish) of size k by replacing at most one vertex by K_2 or S_2. Note that the new vertices result on true or false twins, respectively, and they are in the same set of the new p-partition (C^*, S^*). The elements of S^* are called the *legs* and C^* is called the *body* of the quasi-starfish or quasi-urchin.

Note that there are five possible quasi-starfishes of size two, and they are also the five possible quasi-urchins of size two: P_4, P, $\overline{P}$, fork and kite (see Figure 1). To avoid ambiguity, we will consider these five graphs as quasi-starfishes, while quasi-urchins will be always of size at least three.

When considering quasi-urchins and quasi-starfishes, we have ten kinds of them. We will call *type 1* (resp. *type 2*) the urchins (resp. starfishes); *type 3* (resp. *type 4*) the urchins (resp. starfishes), where a vertex in the body was replaced by K_2; *type 5* (resp. *type 6*) the urchins (resp. starfishes), where a vertex in the body was replaced by S_2; *type 7* (resp. *type 8*) the urchins (resp. starfishes), where a leg was replaced by K_2; and *type 9* (resp. *type 10*) the urchins (resp. starfishes), where a leg was replaced by S_2. Recall that graphs of odd type have always size at least three and, with this condition, the ten types form a partition over the family of quasi-urchins and quasi-starfishes.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$, such that G_1 is p-separable with partition (V_1^1, V_1^2). Consider the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{xy \mid x \in V_1^1, y \in V_2\}$. We shall denote this graph by $G_1 \veebar G_2$.

Theorem 4. [20] *Every graph G either is p-connected or can be obtained uniquely from its p-components and weak vertices by a finite sequence of $\cup$, $\vee$ and $\veebar$ operations.*

Proposition 1. [10]*A graph G is P_4-tidy if and only if every p-component is isomorphic to either P_5 or $\overline{P_5}$ or C_5 or a quasi-starfish or a quasi-urchin. Quasi-starfishes and quasi-urchins are the p-separable p-components of G.*

Remark 1. *Let G_1 be a quasi-urchin or a quasi-starfish, and G_2 be a graph. If G_1 is type 7 or 8, all the legs are simplicial vertices both in G_1 and in $G_1 \veebar G_2$. Otherwise, both in G_1 and in $G_1 \veebar G_2$, each leg of G_1 is controlled by a vertex in the body of G_1. Then, by Lemmas 2 and 3, for every coloring of G_1 (resp. $G_1 \veebar G_2$) with more than $\chi(G_1)$ (resp. $\chi(G_1 \veebar G_2)$) colors, if there is a dominant vertex of color c in $V(G_1)$, then there is a dominant vertex of color c in the body of G_1.*

Lemma 4. *Let G be a quasi-starfish or quasi-urchin of size k. Then,*

i. If G is type 1,2,5,6,7,9 or 10, then $\chi(G) = k$.

ii. if G is type 3,4 or 8, then $\chi(G) = k + 1$.

iii. $\chi_b(G) = \chi(G)$.

Proof. Items *i.* and *ii.* are easy to prove, since a coloring of the maximum clique of G can be extended to the whole graph without increasing the number of colors. Let (C^*, S^*) be the p-partition of G. To prove item *iii.*, suppose on the contrary that we have a b-coloring φ of G with more than $\chi(G)$ colors. By Remark 1, if there is a dominant vertex of color c in G, then there is a dominant vertex of color c in C^*. If G is neither type 5 nor type 6, then $|C^*| \leq \chi(G)$, a contradiction. If G is type 5 or 6, then there is a pair of false twins in C^*, so by Lemma 1, at most $|C^*| - 1$ different colors can have dominant vertices and $|C^*| - 1 \leq \chi(G)$, a contradiction. □

Lemma 5. *Let $G_1 = (V_1, E_1)$ be a p-separable P_4-tidy graph, and $G_2 = (V_2, E_2)$ a graph such that $V_1 \cap V_2 = \emptyset$. Then,*

i. If G_1 is not type 8, then $\chi(G_1 \veebar G_2) = \chi(G_1) + \chi(G_2)$; if G_1 is type 8, then $\chi(G_1 \veebar G_2) = \chi(G_1) + \chi(G_2) - 1$.

ii. If G_1 is not type 8, then $\chi_b(G_1 \veebar G_2) = \chi_b(G_1) + \chi_b(G_2)$; if G_1 is type 8, then $\chi_b(G_1 \veebar G_2) = \chi_b(G_1) + \chi_b(G_2) - 1$.

Proof. Let $G = G_1 \veebar G_2$. By Proposition 1, G_1 is a quasi-urchin or a quasi-starfish. Let (C^*, S^*) be its p-partition. Then G contains $G_1[C^*] \vee G_2$ as an induced subgraph, thus $\chi(G) \geq \chi(G_1[C^*] \vee G_2)$. On the other hand, every coloring of $G_1[C^*] \vee G_2$ can be extended to G without adding new colors, by giving to each vertex in S^* either a color used by a non-neighbor of it in C^* or a color used in G_2. Hence, $\chi(G) = \chi(G_1[C^*] \vee G_2)$. By Theorem 2, $\chi(G_1[C^*] \vee G_2) = \chi(G_1[C^*]) + \chi(G_2)$. By Lemma 4, if G_1 is type 8 then $\chi(G_1[C^*]) = \chi(G_1) - 1$, otherwise $\chi(G_1[C^*]) = \chi(G_1)$. This concludes the proof of item *i.*.

In order to prove item *ii.*, we will show that $\chi_b(G) = \chi_b(G_2) + \chi(G_1[C^*])$. Any b-coloring of G_2 can be extended to a b-coloring of G by assigning $\chi(G_1[C^*])$ new colors to C^* and giving to each vertex in S^* either a color used by a non-neighbor of it in C^* or a color used in G_2. So, $\chi_b(G) \geq \chi_b(G_2) + \chi(G_1[C^*])$.

If $\chi_b(G) = \chi(G)$, by item *i.*, $\chi_b(G) = \chi(G_2) + \chi(G_1[C^*]) \leq \chi_b(G_2) + \chi(G_1[C^*])$. So, we may suppose $\chi_b(G) > \chi(G)$. Let now φ be a b-coloring of G with more than $\chi(G)$ colors. By Remark 1, if there is a dominant vertex of color c in G, then there is a dominant vertex of color c in $C^* \cup V(G_2)$. Notice that the set of colors used by vertices in G_2 and the set of colors used in C^* are disjoint, so C^* should contain dominant vertices for all the colors used in $V(C^*)$. In particular, if G_1 is type 5 or 6, by Lemma 1, it follows that the two non-adjacent vertices in C^* receive the same color, thus C^* is colored with $\chi(G_1[C^*])$ colors. On the other hand, it is easy to see that φ restricted to $V(G_2)$ is a b-coloring of G_2. So $\chi_b(G) \leq \chi_b(G_2) + \chi(G_1[C^*])$.

We have proved that $\chi_b(G) = \chi_b(G_2) + \chi(G_1[C^*])$. By Lemma 4, if G is type 8 then $\chi(G_1[C^*]) = \chi(G_1) - 1 = \chi_b(G_1) - 1$, otherwise $\chi(G_1[C^*]) = \chi(G_1) = \chi_b(G_1)$. This concludes the proof of item *ii.*. □

2. b-continuity in P_4-tidy graphs

In [3], a family of cographs with arbitrarily large difference between their b-chromatic number and their chromatic number was shown. Therefore, it makes sense to analyze b-continuity in P_4-tidy graphs. In this section we prove that P_4-tidy graphs are b-continuous, by using the decomposition theorem for this class of graphs.

Lemma 6. *If G is P_5, $\overline{P_5}$, C_5, a quasi-urchin or a quasi-starfish, then G is b-continuous.*

Proof. If $G = P_5$, then $\chi(G) = 2$ and $\chi_b(G) = 3$ and, for the remaining cases, by Lemma 4, $\chi_b(G) = \chi(G)$. So, they are trivially b-continuous. □

Lemma 7. [3] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \cup G_2$, then G is b-continuous.*

Lemma 8. [3] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \vee G_2$, then G is b-continuous.*

Lemma 9. *Let $G_1 = (V_1, E_1)$ be a p-separable P_4-tidy graph and $G_2 = (V_2, E_2)$ be a graph such that $V_1 \cap V_2 = \emptyset$. If G_2 is b-continuous and $G = G_1 \veebar G_2$, then G is b-continuous.*

Proof. By Proposition 1, G_1 is a quasi-starfish or a quasi-urchin. Let (C^*, S^*) be the p-partition of G_1. Suppose first that G_1 is not type 8. Any b-coloring of G_2 with t colors $\{1, \ldots, t\}$ can be extended to a b-coloring of G with $t + \chi(G_1)$ colors, in the following way. If we color G_1 using colors $\{t+1, \ldots, t+\chi(G_1)\}$, then every dominant vertex in G_2 will have now also neighbors with colors $t+1, \ldots, t+\chi(G_1)$, and every dominant vertex in C^* will have now also neighbors with colors $1, \ldots, t$. Since C^* contains dominant vertices of all colors in $\{t+1, \ldots, t+\chi(G_1)\}$, the resulting coloring is a b-coloring of G with $t+\chi(G_1)$ colors.

Suppose now that G_1 is type 8. Any b-coloring of G_2 with t colors $\{1, \ldots, t\}$ can be extended to a b-coloring of G with $t + \chi(G_1) - 1$ colors, in the following way. If we color G_1 using colors $\{t, \ldots, t + \chi(G_1) - 1\}$ in such a way that C^* uses colors from $t + 1$ to $t + \chi(G_1) - 1$, then every dominant vertex in G_2 will have now also neighbors with colors $t+1, \ldots, t+\chi(G_1)-1$, and every dominant vertex in C^* will have now also neighbors with colors $1, \ldots, t$. Since C^* contains dominant vertices of all colors in $\{t+1, \ldots, t+\chi(G_1)-1\}$, this results in a b-coloring of G with $t + \chi(G_1) - 1$ colors.

Since G_2 is b-continuous, we can obtain b-colorings for G with each color t', where $\chi(G_2) + \chi(G_1) \leq t' \leq \chi_b(G_2) + \chi_b(G_1)$ in the first case, and $\chi(G_2) + \chi(G_1) - 1 \leq t' \leq \chi_b(G_2) + \chi_b(G_1) - 1$ in the second case. By Lemma 5, $\chi(G) = \chi(G_2) + \chi(G_1)$ and $\chi_b(G) = \chi_b(G_2) + \chi_b(G_1)$ in the first case, while $\chi(G) = \chi(G_2) + \chi(G_1) - 1$ and $\chi_b(G) = \chi_b(G_2) + \chi_b(G_1) - 1$ in the second case, so G is b-continuous. □

Theorem 5. *P_4-tidy graphs are b-continuous.*

Proof. Immediate by an inductive argument using the decomposition Theorem 4, Proposition 1, Lemmas 7, 8 and 9 and Lemma 6 for the base case of the induction. □

3. Computation of the b-chromatic number in P_4-tidy graphs

The inequality in part *i.* of Theorem 3 can be strict, and this fact prevents us from using this result for directly computing the b-chromatic number of P_4-tidy graphs by using the decomposition Theorem 4. In fact, it is not difficult to build examples showing that the b-chromatic number of the graph $G_1 \cup G_2$ does not depend only on the b-chromatic numbers of G_1 and G_2. To overcome this problem, we follow the approach in [3] in the definition of the *dominance sequence* $dom_G \in \mathbb{Z}^{\mathbb{N}_{\geq \chi(G)}}$ of a graph G, where $dom[t]$ is the maximum number of distinct color classes that admit dominant vertices in any coloring of G with t colors, for $\chi(G) \leq t \leq |V(G)|$. We will compute this sequence recursively on P_4-tidy graphs by using the decomposition theorem. Then we will obtain the b-chromatic number of G as the maximum t such that $dom_G[t] = t$.

Lemma 10. *Let G be P_5, $\overline{P_5}$, C_5, a quasi-urchin or a quasi-starfish. The dominance sequence for G can be obtained in linear time.*

Proof. It is easy to see that $dom_{P_5}[2] = 2$, $dom_{P_5}[3] = 3$, and $dom_{P_5}[t] = 0$ for $t \geq 4$; $dom_{\overline{P_5}}[3] = 3$, $dom_{\overline{P_5}}[4] = 1$, and $dom_{\overline{P_5}}[5] = 0$; $dom_{C_5}[3] = 3$, and $dom_{C_5}[t] = 0$ for $t \geq 4$. Now, let $G = (C^*, S^*)$ be a quasi-urchin or quasi-starfish of size k. Let (C, S) be the p-partition of the urchin or starfish, $S = \{s_1, \ldots, s_k\}$, $C = \{c_1, \ldots, c_k\}$. If a vertex in S (resp. C) was replaced by two vertices, we will assume that the vertex was s_1 (resp. c_1) and that it was replaced by vertices s_1', s_1'' (resp. c_1', c_1''). Recall that, for every graph G, $dom_G[\chi(G)] = \chi(G)$. Consider now colorings of G with more than $\chi(G)$ colors. By Remark 1, if there is a dominant vertex of color c in G, then there is a dominant vertex of color c in C^*. So, for $t > \chi(G)$, $dom_G[t] \leq |C^*|$.

If G is type 1, then $dom_G[k] = dom_G[k+1] = k$ and $dom_G[t] = 0$ for $t \geq k+2$; if G is type 2, then $dom_G[k+s] = \min\{k, 2k-2s\}$ for $0 \leq s \leq k$, and $dom_G[t] = 0$ for $t > 2k$ [3].

We start by analyzing the different kinds of quasi-urchins.

Claim 1. If G is type 3, then $dom_G[k+1] = dom_G[k+2] = k+1$, $dom_G[t] = 0$ for $t \geq k+3$.

In G there are $k+1$ vertices of degree $k+1$ and no vertex of degree at least $k+2$, so the upper bounds for each value of dom_G are clear (a dominant vertex in a coloring with t colors must have degree at least $t-1$). A coloring with $k+2$ colors and $k+1$ dominant vertices of different colors can be obtained by coloring all the vertices in S^* with the same color, different from the colors used in C^*. $\diamondsuit$

Claim 2. If G is type 5, then $dom_G[k] = dom_G[k+1] = k$, $dom_G[k+2] = k-1$, $dom_G[t] = 0$ for $t \geq k+3$.

Since $k \geq 3$, in G there are $k-1$ vertices of degree $k+1$, 2 vertices of degree k, and no vertex of degree at least $k+2$. So, the upper bounds on $dom_G[t]$ for $t \geq k+2$ are clear. The upper bound for $dom_G[k+1]$ holds by Lemma 1. Two colorings attaining the upper bounds for $dom_G[k+1]$ and $dom_G[k+2]$ are defined as follows. Vertices $c_2, \ldots, c_k$ receive colors $1, \ldots, k-1$; vertices $s_1, \ldots, s_k$ receive color $k+1$; vertices c'_1, c''_1 receive both color k or colors k and $k+2$, respectively. $\diamondsuit$

Claim 3. If G is type 7 or type 9, then $dom_G[k] = dom_G[k+1] = k$, $dom_G[k+2] = 1$, $dom_G[t] = 0$ for $t \geq k+3$.

Since $k \geq 3$, in G there are $k-1$ vertices of degree k, one vertex of degree $k+1$, and no vertex of degree at least $k+2$. So, the upper bounds on $dom_G[t]$ are clear. Two colorings attaining the upper bounds for $dom_G[k+1]$ and $dom_G[k+2]$ are defined as follows. Vertices $c_1, \ldots, c_k$ receive colors $1, \ldots, k$; vertices $s'_1, s_2, \ldots, s_k$ receive color $k+1$; vertex s''_1 receives color 2 or $k+2$, respectively. $\diamondsuit$

We will now analyze the different kinds of quasi-starfishes.

Claim 4. If G is type 4, then $dom_G[k+1+s] = \min\{k, 2k-2s\} + 1$ for $0 \leq s < k$, and $dom_G[t] = 0$ for $t > 2k$.

Since $\chi(G) = k+1$, then $dom_G[k+1] = k+1$. Since the maximum degree of G is $2k-1$, it is clear that $dom_G[t] = 0$ for $t > 2k$. Let $t = k+1+s$ such that $1 \leq s \leq k-1$ and let φ be a coloring of G with t colors and maximum number of colors with dominant vertices. At least one of c'_1, c''_1 has a color different from $\varphi(s_1)$. Suppose without loss of generality that $\varphi(c''_1) \neq \varphi(s_1)$, then $\varphi(c''_1) \neq \varphi(v)$ for every $v \in V(G)$. Let $G' = G - \{c''_1\}$. Thus the restriction of φ to G' is a coloring with $t-1$ colors, and dominant vertices of G are still dominant in G', therefore $dom_G[t] \leq dom_{G'}[t-1] + 1$. Conversely, let ψ be a coloring of G' with $t-1$ colors (namely, colors $1, \ldots, t-1$) and maximum number of colors with dominant vertices. We can extend ψ to a t-coloring of G by defining $\psi(c''_1) = t$. Since $t-1 \geq k+1$, no vertex in S^* was dominant in G', so every dominant vertex of G' is still dominant in G. Besides, c''_1 is now dominant in G if and only if $\psi(s_1) = \psi(v)$ for some vertex v of G', different from s_1, and this happens if and only if c'_1 was dominant in G'. By symmetry of G', we may assume that if $dom_{G'}[t-1] > 0$ then c'_1 was dominant in G'. So, if $dom_{G'}[t-1] > 0$, we have that $dom_G[t] = dom_{G'}[t-1] + 1$. Since G' is type 2, we already know that $dom_{G'}[k+s] = \min\{k, 2k-2s\}$. Since $s \leq k-1$, $dom_{G'}[t-1] > 0$, and $dom_G[k+1+s] = \min\{k, 2k-2s\} + 1$. $\diamondsuit$

Claim 5. If G is type 6, then $dom_G[k+s] = k$ for $0 \leq s \leq \lfloor \frac{k}{2} \rfloor$, $dom_G[k+s] = \min\{k-1, 2k-2s+2\}$ for $\lfloor \frac{k}{2} \rfloor \leq s \leq k$, and $dom_G[k+s] = 0$ for $s > k$.

Since $\chi(G) = k$, then $dom_G[k] = k$. Since the maximum degree of G is $2k-1$, it is clear that $dom_G[t] = 0$ for $t > 2k$. Let $t = k+s$ with $1 \leq s \leq k$ and let φ be a coloring of G with t colors and maximum number of colors with dominant vertices.

Suppose first that $\varphi(c'_1) = \varphi(c''_1)$. Then the number of colors with dominant vertices in G is the same as the number of colors with dominant vertices when restricting φ to

$G' = G - \{c_1''\}$. Conversely, any coloring of G' can be extended to a coloring of G by giving to c_1'' the color used by c_1', thus preserving the dominant vertices. Then, if $\varphi(c_1') = \varphi(c_1'')$, it follows that $dom_G[k+s] = dom_{G'}[k+s]$ and, since G' is type 2, $dom_G[k+s] = \min\{k, 2k-2s\}$.

Suppose now that $\varphi(c_1') \neq \varphi(c_1'')$. By Lemma 1, none of c_1', c_1'' is dominant. So, in this case, the number of colors with dominant vertices is at most $k-1$. We may assume $2s > k$, otherwise, by the arguments above, we can find a coloring φ' of G with $\varphi'(c_1') = \varphi'(c_1'')$ and such that there are k colors with dominant vertices. Since $k \geq 2$, this implies $s > 1$, hence $t > k+1$. Since $\varphi(c_1') \neq \varphi(c_1'')$, at least one of them has a color different from $\varphi(s_1)$. Suppose without loss of generality that $\varphi(c_1'') \neq \varphi(s_1)$, then $\varphi(c_1'') \neq \varphi(v)$ for every $v \in V(G)$. Let $G' = G - \{c_1''\}$. Thus the restriction of φ to G' is a coloring with $t-1$ colors, and dominant vertices of G are still dominant in G'. Since c_1'' was not dominant in G, the number of colors with dominant vertices does not decrease. Conversely, let ψ be a coloring of G' with $t-1$ colors (namely, colors $1, \dots, t-1$) and maximum number of colors with dominant vertices. By Lemma 3, all the dominant vertices are in C^*. We can extend ψ to a t-coloring of G by defining $\psi(c_1'') = t$. All dominant vertices in $\{c_2, \dots, c_k\}$ are still dominant. If there were less than k dominant vertices, we may assume by symmetry of G' that they were in $\{c_2, \dots, c_k\}$. If there were k dominant vertices in G', vertex c_1 is no longer dominant, still $c_2, \dots, c_k$ are dominant, and we know that, if $\varphi(c_1') \neq \varphi(c_1'')$, then in G there cannot be more than $k-1$ colors with dominant vertices. So, in that case, $dom_G[k+s] = \min\{k-1, dom_{G'}[k+s-1]\}$. Since G' is type 2, $dom_G[k+s] = \min\{k-1, 2k-2s+2\}$.

So, if $2s \leq k$, then $dom_G[k+s] = k$ and the optimum is attained by a coloring where c_1' and c_1'' receive the same color. If $2s > k$, then $dom_G[k+s] = \min\{k-1, 2k-2s+2\}$ and the optimum is attained by a coloring where c_1' and c_1'' receive different colors. ♢

Claim 6. If G is type 8, then $dom_G[k+1] = k+1$; then $dom_G[k+s] = k$ for $2 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$; $dom_G[k+s] = k-1$ for $s = \frac{k+2}{2}$ (when k is even); $dom_G[k+s] = 2k-2s+2$ for $\lfloor \frac{k+3}{2} \rfloor \leq s \leq k$; and $dom_G[t] = 0$ for $t > 2k$.

Since $\chi(G) = k+1$, then $dom_G[k+1] = k+1$. Since the maximum degree of G is $2k-1$, it is clear that $dom_G[t] = 0$ for $t > 2k$. Let $t = k+s$ with $2 \leq s \leq k$ and let φ be a t-coloring of G with maximum number of colors with dominant vertices. For $i \geq 2$, vertex c_i will be dominant if and only if color $\varphi(s_i)$ is used by some other vertex in G, and vertex c_1 will be dominant if and only if colors $\varphi(s_1')$ and $\varphi(s_1'')$ are used by some other vertices in $(c_1, s_2, \dots, s_k)$. We may assume without loss of generality that $\varphi(c_i) = i$, for $i = 1, \dots, k$, and that vertices in S^* use colors $k+1, \dots, k+s$. If some vertex s_i uses a color at most k, we can always recolor it with a color from $k+1, \dots, k+s$ that is already used in S^*. Since $s \geq 2$, we can do it also for s_1' and s_1''. If $2s \leq k+1$, we can assign colors $k+1, \dots, k+s$ to vertices in S^*, repeating each of them at least once, in such a way that all the vertices in C^* are dominant. If $2s > k+1$, this is not possible. Since $\varphi(s_1') \neq \varphi(s_1'')$ and all the colors $k+1, \dots, k+s$ are used in S^*, we may assume without loss of generality that $\varphi(s_1') = k+1$, $\varphi(s_1'') = k+2$, and $\varphi(s_i) = k+1+i$ for $i = 2, \dots, s-1$ (when $s \geq 3$). To each of the $k+1-s$ remaining vertices we can assign different colors from $k+1, \dots, k+s$. If we assign color $k+1+i$ to vertex s_j, with $s \leq j \leq k$ and $2 \leq i \leq s-1$, both c_i and c_j

become dominant. If we assign color $k+1$ (resp. $k+2$) to some vertex s_j with $s \leq j \leq k$, then c_j will be dominant but c_1 will be dominant only if some other vertex $s_{j'}$, $s \leq j' \leq k$, receives $k+2$ (resp. $k+1$). So, as we have less than s remaining vertices, the optimum $2(k+1-s)$ is attained by assigning to $s_s, \ldots, s_k$ different colors from $k+3$ to $k+s$ when $k+1-s \leq s-2$. The last case is when $k+1-s = s-1$, that is, k is even and $2s = k+2$. In this case we can assign to $s_s, \ldots, s_{k-1}$ different colors from $k+3$ to $k+s$ and to vertex s_k color $k+1$. In this case, all the vertices of C^* but c_1 are dominant, and this is optimal. $\diamondsuit$

Claim 7. If G is type 10, then $dom_G[k+s] = k$ for $0 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$; $dom_G[k+s] = k-1$ for $s = \frac{k+2}{2}$ (when k is even); $dom_G[k+s] = 2k-2s+2$ for $\lfloor \frac{k+3}{2} \rfloor \leq s \leq k$; and $dom_G[t] = 0$ for $t > 2k$.

Since $\chi(G) = k$, then $dom_G[k] = k$. A coloring with $k+1$ colors and k dominant vertices is obtained by giving colors $1, \ldots, k$ to vertices in C^* and color $k+1$ to each vertex in S^*. Since the maximum degree of G is $2k-1$, it is clear that $dom_G[t] = 0$ for $t > 2k$. The arguments for $k+2 \leq s \leq 2k$ are very similar to those in the proof of Claim 6, and are omitted. $\diamondsuit$

In all the cases, given the type of the graph, the dominance sequence can be computed in linear time. The type of the graph can be also determined in linear time [10]. $\square$

Theorem 6. [3] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Let $G = G_1 \cup G_2$ and $t \geq \chi(G)$. Then*

$$\mathrm{dom}_G[t] = min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$$

Theorem 7. [3] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Let $G = G_1 \vee G_2$ and $\chi(G) \leq t \leq |V(G)|$. Let $a = max\{\chi(G_1), t - |V(G_2)|\}$ and $b = min\{|V(G_1)|, t - \chi(G_2)\}$. Then $a \leq b$ and*

$$\mathrm{dom}_G[t] = max_{a \leq j \leq b}\{\mathrm{dom}_{G_1}[j] + \mathrm{dom}_{G_2}[t-j]\}$$

Theorem 8. *Let $G_1 = (V_1, E_1)$ be a quasi-urchin or a quasi-starfish of size k and $G_2 = (V_2, E_2)$ be a graph such that $V_1 \cap V_2 = \emptyset$, $V_2 \neq \emptyset$. Let $G = G_1 \veebar G_2$. Then, the following statements hold.*

i. If G_1 is type 1,2,7,9 or 10, then
- *a.* $\mathrm{dom}_G[k+r] = k + \mathrm{dom}_{G_2}[r]$, *for* $\chi(G_2) \leq r \leq |V_2|$;
- *b.* $\mathrm{dom}_G[k+|V_2|+s] = \mathrm{dom}_{G_1}[k+s]$, *for* $1 \leq s \leq |V_1| - k$.

ii. If G_1 is type 3 or 4, then
- *a.* $\mathrm{dom}_G[k+1+r] = k+1+\mathrm{dom}_{G_2}[r]$, *for* $\chi(G_2) \leq r \leq |V_2|$;
- *b.* $\mathrm{dom}_G[k+1+|V_2|+s] = \mathrm{dom}_{G_1}[k+1+s]$, *for* $1 \leq s \leq |V_1| - k - 1$.

iii. If G_1 is type 5 or 6, then
- *a.* $\mathrm{dom}_G[k+\chi(G_2)] = k + \chi(G_2)$;
- *b.* $\mathrm{dom}_G[k+r] = \max\{k+\mathrm{dom}_{G_2}[r], k-1+\mathrm{dom}_{G_2}[r-1]\}$, *for* $\chi(G_2) < r \leq |V_2|$;
- *c.* $\mathrm{dom}_G[k+1+|V_2|] = \max\{k, k-1+\mathrm{dom}_{G_2}[|V_2|]\}$;
- *d.* $\mathrm{dom}_G[k+|V_2|+s] = \mathrm{dom}_{G_1}[k+s]$, *for* $2 \leq s \leq |V_1| - k$.

iv. If G_1 is type 8, then

a. $\mathrm{dom}_G[k+r] = k + \mathrm{dom}_{G_2}[r]$, *for* $\chi(G_2) \le r \le |V_2|$*;*

b. $\mathrm{dom}_G[k+1+|V_2|] = k$*;*

c. $\mathrm{dom}_G[k+|V_2|+s] = \mathrm{dom}_{G_1}[k+s]$, *for* $2 \le s \le |V_1| - k$*.*

Proof. Recall that $dom[\chi(G)] = \chi(G)$, and that the chromatic number of each type of quasi-starfish or quasi-urchin is described in Lemma 4. Let (C^*, S^*) be the p-partition of G_1. Notice first that in any coloring of G, the set of colors used by V_2 and C^* are disjoint. Let φ be a coloring of G with t colors, $t > \chi(G)$. Vertices in S^* are either simplicial or have degree at most $\chi(G) - 1$ (recall that $V_2 \neq \emptyset$). So no vertex in S^* can be dominant. If some vertex of S^* has a color that is used neither in V_2 nor in C^*, then no vertex in V_2 is dominant. We start the case analysis. If G_1 is type 1,2,7,9 or 10, then C^* is a clique of size k. Every vertex in C^* is dominant when the colors used by S^* are used also in $C^* \cup V_2$, and they are still dominant if we consider φ restricted to $G[V_2 \cup C^*]$. By Theorem 7, $dom_G[k+r] = k + dom_{G_2}[r]$, for $\chi(G_2) \le r \le |V_2|$. If $t > k + |V_2|$, at least some color must be used only in S^*. So the only candidates to be dominant vertices are vertices in C^*. Since they are adjacent to all the vertices in V_2, we may assume that no vertex in S^* uses a color used in V_2, and each vertex of C^* is dominant if and only if it is dominant in $G[V_1]$, so $dom_G[k+|V_2|+s] = dom_{G_1}[k+s]$, for $1 \le s \le |V_1| - k$ $(*)$. If G_1 is type 3 or 4, the analysis is the same but taking into account that C^* is a clique of size $k+1$. If G_1 is type 5 or 6, then C^* is not a clique. We may assume that the original set was $C = \{c_1, \ldots, c_k\}$ and vertex c_1 was replaced by two false twins c_1', c_1''. Item *iii.a* holds because $\chi(G) = k + \chi(G_2)$. Most of the observations for the previous cases still hold. So, when $\chi(G_2) < t - k \le |V_2| + 1$, we have two possibilities to color C^*: we can either use k colors, being $\varphi(c_1') = \varphi(c_1'')$, and in that case k vertices of different colors will be dominant in C^*, or use $k+1$ colors and, by Lemma 1, only $k-1$ vertices in C^* will be dominant. This leads to the expressions *iii.b* and *iii.c*. Finally, when $t > k + |V_2| + 1$, at least some color must be used only in S^*. The analysis in $(*)$ leads to the expression *iii.d*. Finally, if G_1 is type 8, then C^* is a clique of size k but $\chi(G_1) = k+1$. In this case, if $\chi(G_2) \le r \le |V_2|$, necessarily one color in V_2 will be used also in S^*, but the analysis is the same as in case *i.a*. Also the case *iv.c* is similar to *i.b*. The only difference is when $t = k + 1 + |V_2|$. We cannot say that $dom_G[k+1+|V_2|] = dom_{G_1}[k+1] = k+1$, because we know that we have dominant vertices only in C^*, so $dom_G[k+1+|V_2|] \le k$. A coloring with k dominant vertices in C^* is attainable by giving colors $1, \ldots, k$ to vertices in C^*, color $k+1$ to vertices in $S^* \setminus \{s_1''\}$, color $k+2$ to s_1'', and colors $k+2, \ldots, k+1+|V_2|$ to vertices in V_2. □

Theorem 9. *The dominance vector and the b-chromatic number of a P_4-tidy graph G can be computed in $O(n^3)$ time.*

Proof. The previous results give a dynamic programming algorithm to compute the dominance sequence of a P_4-tidy graph from its decomposition tree, that can be computed in linear time [10]. By Theorem 6, Theorem 7, Theorem 8, Theorem 4, Proposition 1 and the fact that P_4-tidy graphs are hereditary, we can compute recursively the dominance vector and consequently the b-chromatic number of G in $O(n^3)$ time. Indeed, if $G = G_1 \cup G_2$, by Theorem 6, the value for $dom_G[t]$ is obtained from $dom_{G_1}[t]$ and $dom_{G_2}[t]$ directly. By Theorem 8, the same case holds for $G = G_1 \veebar G_2$. If $G = G_1 \vee G_2$, we must examine at most n values of j for each value of t, by Theorem 7. We have at most n of these reduction steps, because in each case we must compute two disjoint subgraphs. The base case, computing the dominance sequence of the trivial graph and the five elementary subgraphs in the decomposition, can be done in $O(1)$ by Lemma 10. So the total computation time is $O(n^3)$. Once we have computed the dominance sequence of G, we obtain the b-chromatic number as the maximum value t such that $dom_G[t] = t$. □

4. b-monotonicity in P_4-tidy graphs

In this section, we will show that P_4-tidy graphs are b-monotonic. To this end, we will prove the following property.

Theorem 10. *For every P_4-tidy graph G, every induced subgraph H of G and every $t \geq \chi(G)$, $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ holds.*

We first state some necessary results.

Lemma 11. *Let G be a P_5, a $\overline{P_5}$, a C_5, a quasi-urchin or a quasi-starfish. Then, for every $t \geq \chi(G)$ and every vertex v of G, $\mathrm{dom}_{G-\{v\}}[t] \leq \mathrm{dom}_G[t]$ holds.*

Proof. The cases P_5, $\overline{P_5}$ and C_5 are easy to verify. Let $G = (C^*, S^*)$ be a quasi-urchin or quasi-starfish of size k. Let (C, S) be the p-partition of the urchin or starfish, $S = \{s_1, \ldots, s_k\}$, $C = \{c_1, \ldots, c_k\}$. If a vertex in S (resp. C) was replaced by two vertices, we will assume that the vertex was s_1 (resp. c_1) and that it was replaced by vertices s'_1, s''_1 (resp. c'_1, c''_1). Let $t \geq \chi(G)$, and let v be a vertex of G. Let φ be a t-coloring of $G - \{v\}$ that maximizes the number of color classes with dominant vertices. Suppose first that v is a leg of G and either G is not type 8 or v is different from s'_1, s''_1. Then φ can be extended to a t-coloring of G with the same number of dominant vertices by giving to v the color of some vertex in the body non-adjacent to it. If G is type 8 and $v = s'_1$, since $t \geq \chi(G) = k + 1$, we can give to s'_1 either a color that is not used in the body of G or the color $\varphi(c_1)$ (depending on whether $\varphi(s''_1) = \varphi(c_1)$ or not). Now, suppose that v is a vertex in the body of G. If v has a false twin, we can color v with the color used by its false twin. Otherwise, since $t \geq \chi(G)$, there is some color c that is not used in the body of G. We will extend φ to a t-coloring of G with at least the same number of dominant vertices by setting $\varphi(v) = c$. If some leg w of G adjacent to v was colored c, then all its neighbors are also neighbors of v, so we can recolor w with the color of some vertex in the body non-adjacent to it, and all dominant vertices will still be dominant. The only case in which we cannot do this is when G is type 8, v is not c_1, one of s'_1, s''_1 uses color c and the other one uses color $\varphi(c_1)$. But, in that case, since $t \geq \chi(G) = k + 1$, there are in fact at least two colors c, c' not used in the body of G. So we can give color c' to v, and recolor as mentioned above all the legs adjacent to it (note that neither s'_1 nor s''_1 use c' in the case we are dealing with). Hence, $dom_{G-\{v\}}[t] \leq dom_G[t]$. □

Lemma 12. *Let $G_1 = (V_1, E_1)$ be a quasi-starfish or a quasi-urchin and $G_2 = (V_2, E_2)$ be a b-continuous graph such that $V_1 \cap V_2 = \emptyset$ and, for every $t \geq \chi(G_2)$ and every induced subgraph H of G_2, $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_2}[t]$. Let $G = G_1 \veebar G_2$. Then, for every $t \geq \chi(G)$ and every vertex v of G, $\mathrm{dom}_{G-\{v\}}[t] \leq \mathrm{dom}_G[t]$ holds.*

Proof. If $t = \chi(G)$ the statement is clearly true. Let $t > \chi(G)$, and let v be a vertex of G. Let φ be a t-coloring of $G - \{v\}$ that maximizes the number of color classes with dominant vertices. We will extend φ to a t-coloring of G with the same number of color classes with dominant vertices. Let (C^*, S^*) be the p-partition of G_1. Notice that, since $t > \chi(G) \geq \chi(G - \{v\})$, no vertex in S^* is dominant.

Suppose first that v is a vertex of S^*. We can extend φ by giving to v a color not used by any of its neighbors (it is always possible because $t > \chi(G)$).

Suppose now that v is a vertex of V_2. If $|V_2| = 1$ then the lemma holds by Theorem 8 and the claims in the proof of Lemma 10. If $|V_2| > 1$, let r be the number of colors used by $V_2 - \{v\}$ in φ. If $r \geq \chi(G_2)$, since $dom_{G_2}[r] \geq dom_{G_2-\{v\}}[r]$, we can replace φ restricted to $V_2 - \{v\}$ by an r-coloring of G_2 with $dom_{G_2}[r]$ color classes with dominant vertices, thus obtaining a t-coloring of G with at least the same dominant color classes as before. Otherwise, $r = \chi(G_2 - \{v\}) = \chi(G_2) - 1$. Since $t > \chi(G)$, it follows that $t - r \geq \chi(G_1)$. Notice that the equality can hold only if G_1 is type 8. Then we can replace φ restricted to $V_2 - \{v\}$ by an $(r + 1)$-coloring of G_2 and, by Lemma 10, φ restricted to V_1 by a coloring of G_1 with at most $t - r - 1$ new colors and at least the same dominant color classes as before (if G_1 is type 8 and $t - r = \chi(G_1)$, we can assign to one of the true twin vertices in S^* a color used in V_2).

Finally, suppose that v is a vertex in C^*. If v has a false twin v' in $C^* - \{v\}$, we are done by setting $\varphi(v) = \varphi(v')$. If there are two false twins w, w' in $C^* - \{v\}$ using different colors, we can assign to v color $\varphi(w')$ and to w' color $\varphi(w)$ (possibly recoloring in a suitable way vertices in S^*), obtaining a t-coloring of G with at least the same dominant color classes as before. Otherwise, v is adjacent to all vertices in $C^* - \{v\}$ and they are colored with $\chi(G[C^* - \{v\}])$ colors. Let r be the number of colors used by V_2 in φ. If φ restricted to V_2 is not a b-coloring, we can eliminate one color class from V_2 without decreasing the number of color classes with dominant vertices, and give that color to v, that will be adjacent to all the vertices that were dominant, thus obtaining the desired t-coloring for G. If φ restricted to V_2 is a b-coloring and $r > \chi(G_2)$, since G_2 is b-continuous, we can replace φ restricted to V_2 by a b-coloring of G_2 with $r-1$ colors, thus giving the remaining color to v as before. Finally, by Lemma 5 and being $t > \chi(G)$, if $r = \chi(G_2)$ then $t-r \geq \chi(G_1)$. In that case, it is easy to see that we can replace φ restricted to V_1 by a $t-r$-coloring of G_1, maintaining or increasing the number of color classes with dominant vertices, thus obtaining the desired t-coloring of G. □

Lemma 13. [3] *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. Assume that for every $t \geq \chi(G_i)$ and every induced subgraph H of G_i we have $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$, for $i = 1, 2$. Then, for every $t \geq \chi(G)$ and every induced subgraph H of G, $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ holds.*

Lemma 14. [3] *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be two b-continuous graphs such that* $V_1 \cap V_2 = \emptyset$*, and let* $G = G_1 \vee G_2$*. Assume that for every* $t \geq \chi(G_i)$ *and every induced subgraph* H *of* G_i *we have* $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$*, for* $i = 1, 2$*. Then, for every* $t \geq \chi(G)$ *and every induced subgraph* H *of* G*,* $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ *holds.*

Lemma 15. [3] *Let* G *be a graph. The maximum value of* $\mathrm{dom}_G[t]$ *is attained in* $t = \chi_b(G)$*.*

Proof of Theorem 10. Let us consider a minimal counterexample for the theorem, that is, a P_4-tidy graph G and an induced subgraph H of G such that $dom_H[t] > dom_G[t]$ for some $t \geq \chi(G)$, but such that $dom_{H_2}[t] \leq dom_{H_1}[t]$ for every induced subgraph H_1 of H, every induced subgraph H_2 of H_1 and every $t \geq \chi(H_1)$. By Lemmas 13 and 14, G is neither the union nor the join of two smaller graphs. Let $W = V(G) \setminus V(H)$, namely, $W = \{w_1, \ldots, w_s\}$. If we consider the sequence of graphs defined by $G_0 = G$, $G_i = G_{i-1} - \{w_i\}$ for $1 \leq i \leq s$, it turns out that $G_s = H$. Since $dom_H[t] > dom_G[t]$, for some $i \geq 1$, it holds $dom_{G_i}[t] > dom_{G_{i-1}}[t]$. Since the counterexample was minimal, it should be $i = 1$, thus $H = G - \{w\}$ for some vertex w of G. By Lemma 11, Theorem 5 and Lemma 12, Theorem 4 and Proposition 1, such a counterexample does not exist. □

Corollary 1. *P_4-tidy graphs are b-monotonic.*

Proof. Since P_4-tidy graphs are hereditary, it suffices to show that given a P_4-tidy graph G, $\chi_b(G) \geq \chi_b(H)$ for every induced subgraph H of G. Let G be a P_4-tidy graph, and let H be and induced subgraph of G. If $\chi_b(H) < \chi(G)$, then $\chi_b(H) < \chi_b(G)$. Otherwise, by Theorem 10, $\chi_b(H) = dom_H[\chi_b(H)] \leq dom_G[\chi_b(H)]$ and, by Lemma 15, $dom_G[\chi_b(H)] \leq dom_G[\chi_b(G)] = \chi_b(G)$ implying that $\chi_b(G) \geq \chi_b(H)$. □

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