The Logic of Proofs (LP) [1] is a refinement of modal logic introduced by Artemov in 1995 which has recently been proposed for explaining well-known paradoxes arising in the formalization of Epistemic Logic. Assertions of knowledge and belief are accompanied by justifications: the formula \( \vdash A \) implies \( \vdash [t]A \), for some \( t \). This suggests that computational interpretations in terms of the Curry-de Bruijn-Howard isomorphism could yield programming languages supporting a uniform treatment of programs and type derivations. Although some avenues in this direction have been explored (eg. certifying mobile computation [3] and history-aware computation [2]), they have been centred on intuitionistic fragments of LP. This work proposes to extend this analysis to full LP, which is based on classical logic. To this effect, we define the Hypothetical Logic of Proofs (HLP).

The set of formulas and proof witnesses of HLP is defined by the following syntax:

**Formulas:**

\[ A, B ::= P | \bot | A \supset B | [s]A \]

**Proof witnesses:**

\[ r, s, t ::= x^A | v^A | \lambda x^A.s | s \cdot t | !s \ | \text{LET}\, B \vdash v^A \text{BE} r, s \text{ in } t | [\alpha^A]s | (\mu \alpha^A.s) \ | s + t \]

Judgements take the form \( \Theta; \Gamma; \Delta \vdash A \) where \( \Theta = v_1^{B_1}, \ldots, v_m^{B_m}, \Gamma = x_1^{C_1}, \ldots, x_n^{C_n} \) and \( \Delta = \alpha_1^{D_1}, \ldots, \alpha_k^{D_k} \). Hypotheses are represented by three kinds of variables: truth variables (\( x^A \) claims that \( A \) is true), validity variables (\( v^A \) claims that \( A \) is valid), and falsehood variables (\( \alpha^A \) claims that \( A \) is false).

The inference schemes of HLP are:

\[
\begin{align*}
\varM & \quad \Theta;\Gamma;\Delta \vdash A \mid v^A \quad \text{VarM} \\
\var & \quad \Theta;\Gamma;\Delta \vdash A \mid x^A \quad \text{Var} \\
\text{PlusL} & \quad \Theta;\Gamma;\Delta \vdash A \mid s \quad \Theta;\Gamma;\Delta \vdash A \mid t \\
\text{PlusR} & \quad \Theta;\Gamma;\Delta \vdash A \mid s + t \\
\text{Name} & \quad \Theta;\Gamma;\Delta \vdash \alpha^A \mid s \\
\text{NAbs} & \quad \Theta;\Gamma;\Delta \vdash A \mid [\alpha^A]s
\end{align*}
\]

Note that each inference scheme updates the associated proof witness of the inferred judgement. Moreover, this witness is reflected in the logic via \( \Box \). The additional hypothesis \( \Theta;::;::;\vdash s \equiv t: B \) in \( \Box \) is required for subject reduction, which would otherwise fail. This follows from the fact that proof witnesses are reflected in the logic and that normalisation equates them. The schemes PlusL and PlusR correspond to LP’s multiple-conclusion nature (cf. axioms \([s]A \supset [s + t]A\) and \([t]A \supset [s + t]A\) of LP).

We show that HLP can prove all LP-theorems – hence, all classical tautologies – and we provide a translation from HLP to LP which preserves derivability. Also, we define a term assignment, where terms are proof witnesses with additional annotations, and define reduction rules to model proof normalization by means of term reduction. The resulting reduction system extends the \( \lambda \)-calculus, as well as Parigot’s \( \lambda \mu \)-calculus [4]. Finally, we address some fundamental properties of the metatheory, including type preservation, normalisation and confluence.

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**References**