Balancedness of subclasses of circular-arc graphs

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A graph is balanced if its clique-vertex incidence matrix contains no square submatrix of odd order with exactly two ones per row and per column. There is a characterization of balanced graphs by forbidden induced subgraphs, but no characterization by *mininal* forbidden induced subgraphs is known, not even for the case of circular-arc graphs. A circular-arc graph is the intersection graph of a family of arcs on a circle. In this work, we characterize when a given graph G is balanced in terms of minimal forbidden induced subgraphs, by restricting the analysis to the case where G belongs to certain classes of circular-arc graphs, including Helly circular-arc graphs, claw-free circular-arc graphs, and gem-free circular-arc graphs. In the case of gem-free circular-arc graphs, analogous characterizations are derived for two superclasses of balanced graphs: clique-perfect graphs and coordinated graphs.

Keywords: balanced graphs, clique-perfect graphs, circular-arc graphs, coordinated graphs, perfect graphs

1 Introduction

Two fundamental combinatorial optimization problems are set packing and set covering, which can be expressed by

$$\max c^T x \text{ s.t. } Ax \leq \mathbf{1},\ x \in \{0,1\}^n \tag{1}$$

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and

$$\min c^T x \text{ s.t. } Ax \geq \mathbf{1},\ x \in \{0,1\}^n, \tag{2}$$

respectively, where A is some $\{0,1\}$-matrix. The matrix A is *perfect* (resp. *ideal*) if no integrality requirements are needed in (1) (resp. (2)) as the polytope $P(A) = \{x \in \mathbb{R}^n_+ \colon\ Ax \leq \mathbf{1}\}$ (resp. the polyhedron $Q(A) = \{x \in \mathbb{R}^n_+ \colon\ Ax \geq \mathbf{1}\}$) has integral extreme points only. The matrix A is *balanced* if all its submatrices are both perfect and ideal or, equivalently, if it contains no submatrix of odd order with exactly two ones per row and per column [2, 18]. Well-known examples of balanced matrices are totally unimodular matrices where even no integrality requirements are needed in (1) and (2) for varying right hand side vectors.

A graph G is *balanced* if its clique-matrix is balanced. Here, a *clique* Q in a graph $G = (V, E)$ is an inclusion-wise maximal subset of pairwise adjacent vertices and given an enumeration $Q_1, \ldots, Q_k$ of all cliques of G and an order $v_1, \ldots, v_n$ of all vertices of G, a *clique-matrix* of G is the $k \times n$ $\{0,1\}$-matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if and only if $v_j \in Q_i$. The clique-matrix of a graph is unique up to permutations of rows and/or columns. The name 'balanced graphs' appeared explicitly in [3], but these graphs were already considered in [2, see Theorem 5 therein].

The class of balanced graphs is closed under taking induced subgraphs. Examples of balanced graphs are *interval graphs* (the intersection graphs of intervals of a line) and *bipartite graphs* (having a partition of their nodes into two stable sets) as their clique-matrices are totally unimodular [17, 22] and, thus, balanced.

Well-known superclasses of balanced graphs are perfect graphs and hereditary clique-Helly graphs. A graph is *perfect* if its clique-matrix is perfect [15]. Some years ago, the minimal forbidden induced subgraphs of perfect graphs were characterized [14], settling affirmatively a conjecture posed more than 40 years before by Berge [1]. The minimal forbidden induced subgraphs of perfect graphs are the chordless cycles of odd length having at least 5 vertices, called *odd holes* C_{2k+1}, and their complements, the *odd antiholes* $\overline{C_{2k+1}}$.

A graph is *hereditary clique-Helly* if, in any of its induced subgraphs, every nonempty subfamily of pairwise intersecting cliques has a common vertex. It follows from [2] that balanced graphs are hereditary clique-Helly. Prisner [28] characterized hereditary clique-Helly graphs as those graphs containing no induced 0-, 1-, 2-, or 3-pyramid (see Figure 1).

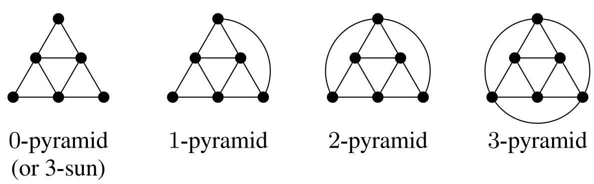

Fig. 1: The pyramids

Hence, no balanced graph contains an odd hole, odd antihole, or any pyramid as induced subgraph. In addition, balanced graphs were characterized by means of forbidden induced subgraphs as follows. For a graph $G = (V, E)$ and $W \subseteq V$, let $N(W) = \bigcap_{w \in W} N(w)$ and use $N(e)$ as shorthand for $N(\{u, v\})$ for an edge $e = uv$, whereas $N(\emptyset) = V(G)$. An *unbalanced cycle* of G is an odd cycle C such that, for each edge $e \in E(C)$ (i.e., joining two consecutive vertices of C), there exists a (possibly empty) complete subgraph W_e of G such that $W_e \subseteq N(e) \setminus V(C)$ and $N(W_e) \cap N(e) \cap V(C) = \emptyset$. Notice that it is possible for the sets W_e and $W_{e'}$ for different edges e and e' to overlap. It is not hard to see that unbalanced cycles are obstructions for balanced graphs. Indeed, if, for each $e \in E(C)$, we choose any clique Q_e of G containing W_e as well as the endpoints of e, then the submatrix M of any clique-matrix of G formed by the rows corresponding to the cliques Q_e for every $e \in E(C)$ and the columns corresponding to the vertices of C has exactly two ones per row and per column. More precisely, for each $e \in E(C)$, the ones in the row of M corresponding to the clique Q_e are exactly in the columns corresponding to the endpoints of e, while the ones in the column of M corresponding to each vertex v of C are exactly in the rows corresponding to the cliques Q_e and $Q_{e'}$ where e and e' are the two edges of C incident to v. An *extended odd sun* is a graph G with an unbalanced cycle C such that $V(G) = V(C) \cup \bigcup_{e \in E(C)} W_e$ and $|W_e| \leq |N(e) \cap V(C)|$ for each edge $e \in E(C)$. The extended odd suns with the smallest number of vertices are C_5 and the pyramids. Moreover, every odd hole is an extended odd sun (by letting $W_e = \emptyset$ for each e). Notice that C_3 is not an extended odd sun, since otherwise we would be forced to choose $W_e = \emptyset$ for each edge e, but then $N(W_e) \cap N(e) \cap V(C) = \{v\}$ where v is the only vertex non-incident to e (because $N(W_e) = N(\emptyset) = V(C)$ and $N(e) = \{v\}$). The characterization of balancedness by forbidden induced subgraphs is the following.

Theorem 1 ([2, 9]) *A graph is balanced if and only if it has no unbalanced cycle, or, equivalently, if and only if it contains no induced extended odd sun.*

However, the above characterization is not by *minimal* forbidden induced subgraphs because some extended odd suns contain some other extended odd suns as induced subgraphs, as Figure 2 shows. Thus, excluding extended odd suns suffices to guarantee balancedness, but it is not necessary to exclude all of them. We address the problem of finding the *minimal forbidden induced subgraphs*, i.e., those graphs that are not balanced but all their proper induced subgraphs are balanced. (An induced subgraph H of a graph G is *proper* if H is different from G.)

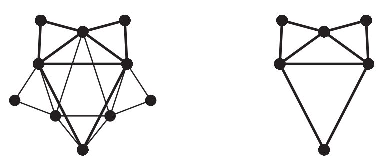

Fig. 2: On the left, an extended odd sun that is not minimal. Bold lines correspond to the edges of a proper induced extended odd sun, depicted on the right.

This problem is still open. Partial answers are obtained in [10] where minimal forbidden induced subgraph characterizations of balanced graphs restricted to the following graph classes are found: P_4-tidy graphs, paw-free graphs, line graphs, and complements of line graphs. In this paper, we study balanced graphs restricted to some subclasses of circular-arc graphs. An extended abstract containing the main results of this work appeared in [11].

Let $\mathcal{F}$ be a family of sets. The *intersection graph* of $\mathcal{F}$ is a graph whose vertices represent the members of $\mathcal{F}$, where two members of $\mathcal{F}$ are adjacent if and only if they intersect. A *circular-arc* (CA) graph is the intersection graph of a family of arcs on a circle [24]. Clearly, CA graphs can be seen as an extension of interval graphs. But while interval graphs form a subclass of balanced graphs, this is not the case for CA graphs. Note that CA graphs are neither perfect nor hereditary clique-Helly in general as odd holes, odd antiholes, and pyramids can be easily seen to be CA graphs. Perfectness of CA graphs was addressed in [32], but the study of balancedness of CA graphs is still in order.

This paper is organized as follows. In Section 2, we present minimal forbidden induced subgraph characterizations of balanced graphs within a superclass of the class of Helly circular-arc graphs and the classes of claw-free circular-arc graphs and gem-free circular-arc graphs. In Section 3, we additionally characterize, within gem-free circular-arc graphs, two further superclasses of balanced graphs: clique-perfect and coordinated graphs (see Section 3 for the definitions).

1.1 Basic definitions

We close this section by providing some basic definitions. All graphs in this paper are undirected, without loops and without multiple edges. Let G be a graph. We denote by $V(G)$ its vertex set, by $E(G)$ its edge set, and by $\overline{G}$ its complement. If $W \subseteq V(G)$, the subgraph *induced* by W in G is denoted by $G[W]$ and is *proper* if $W \neq V(G)$. The subtraction $G - W$ denotes $G[V(G) \setminus W]$. A vertex v is a *cutpoint* of G if $G - \{v\}$ has more connected components than G.

A *universal vertex* of G is adjacent to all the other vertices of G and an *isolated vertex* of G is adjacent to no vertex of G. The *neighborhood* of a vertex v in G consists of all vertices that are adjacent to v and is denoted by $N_G(v)$, or simply $N(v)$ if G is clear from the context. The *common neighborhood of an edge* $e = vw$ is $N_G(e) = N_G(v) \cap N_G(w)$ and, in general, the *common neighborhood of a nonempty set* $W \subseteq V(G)$ is $N_G(W) = \bigcap_{w \in W} N_G(w)$, whereas $N_G(\emptyset) = V(G)$. Two adjacent vertices u and v of G are *true twins* if $N_G(u) \cup \{u\} = N_G(v) \cup \{v\}$.

A *complete* is a set of mutually adjacent vertices. The complete on n vertices will be denoted by K_n. A complete on 3 vertices is said a *triangle*. An inclusion-wise maximal complete is a *clique*. A *stable set* of a graph is a set of pairwise nonadjacent vertices. A set $A \subseteq V(G)$ and a vertex v of $V(G)$ are *complete* to each other if $A \subseteq N_G(v)$, and *anticomplete* if $N_G(v) \cap A = \emptyset$. The set $A \subseteq V(G)$ is *complete* (resp. *anticomplete*) to the set $B \subseteq V(G)$ if A and b are complete (resp. anticomplete) for each $b \in B$. A *dominating set* of G is a set $A \subseteq V(G)$ such that each $v \in V(G) \setminus A$ is adjacent to at least one element of A.

Paths and cycles are assumed to be simple; i.e., with no repeated vertices aside from the starting and ending vertices in the case of cycles. By the *edges* of a cycle C we mean those edges joining two consecutive vertices of C. The set of edges of a cycle C will be denoted by $E(C)$. A *chord* of a cycle (resp. path) is an edge joining two nonconsecutive vertices. We denote the chordless cycle on n vertices by C_n and the chordless path on n vertices by P_n. A chord of a cycle is *short* if its endpoints are at distance two within the cycle, and is *long* otherwise. Two chords ab and cd of a cycle C such that their endpoints are four different vertices of C that appear in the order a, c, b, d in C are called *crossing*. A cycle is *odd* if it has an odd number of vertices, and is called *even* otherwise. A *hole* is a chordless cycle of length at least 4. An *odd antihole* is the complement of an odd hole.

Let G and H be two graphs with $V(G) \cap V(H) = \emptyset$. The *disjoint union* of G and H is a graph $G \cup H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. The disjoint union is clearly an associative operation, and for each nonnegative integer t we will denote by tG the disjoint union of t copies of G. A class $\mathcal{G}$ of graphs is called *hereditary* if, for every graph G of $\mathcal{G}$, each induced subgraph of G belongs to $\mathcal{G}$. The class of balanced graphs is hereditary (see [9]). Let G and H be two graphs. We say that *G is H-free* to mean that G contains no induced H. If $\mathcal{H}$ is a collection of graphs we say that *G is $\mathcal{H}$-free* to mean that G contains no induced H for any $H \in \mathcal{H}$. A graph H is a *forbidden induced subgraph for graph class $\mathcal{C}$* if no graph of $\mathcal{C}$ contains an induced H. Moreover, if $\mathcal{C}$ is a hereditary class,

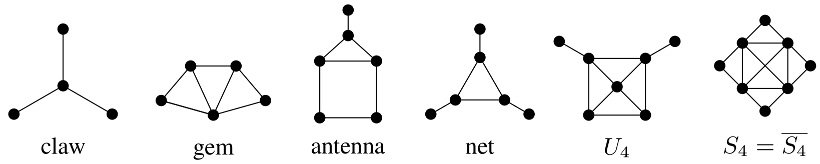

Fig. 3: Some small graphs

H is said a *minimal forbidden induced subgraph for* $\mathcal{C}$ or a *minimally* non-$\mathcal{C}$ *graph* if H does not belong to $\mathcal{C}$ but each proper induced subgraph of H belongs to $\mathcal{C}$. Some small graphs to be referred to in this context are depicted in Figure 3.

For each $t \geq 3$, the *complete* t*-sun* S_t is a graph whose vertex set can be partitioned into a clique $\{q_1, q_2, \ldots, q_t\}$ and a stable set $\{s_1, s_2, \ldots, s_t\}$ such that $N(s_i) = \{q_i, q_{i+1}\}$ for each $i \in \{1, 2, \ldots, t\}$ (where q_{t+1} stands for q_1). The graph S_3 is simply called *3-sun*. The 3-sun is also called *tent* in the literature.

2 Balancedness of some subclasses of circular-arc graphs

Recall that a circular-arc (CA) graph is the intersection graph of a family of open arcs on a circle. Such a family of arcs is called a CA model of the graph. CA graphs were first studied by Tucker [30, 31, 32, 33] and can be recognized in linear time [27]. Some minimal forbidden induced subgraphs for the class of CA graphs are $K_{2,3}$, G_2, G_3, domino, G_5, G_6, $\overline{C_6}$, net $\cup K_1$, G_9, and $C_n \cup K_1$ for each $n \geq 4$ [29] (cf. Figure 4).

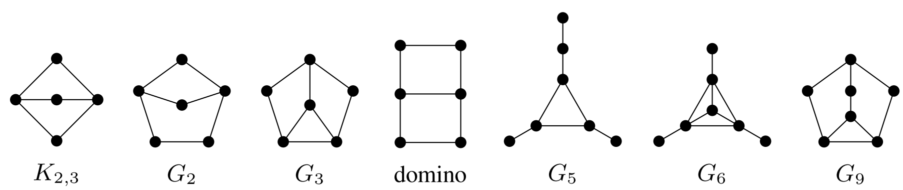

Fig. 4: Some forbidden induced subgraphs for the class of circular-arc graphs

Since $C_n \cup K_1$ is not a CA graph for any $n \geq 4$, if G is a CA graph and H is a hole of G, then $V(H)$ is dominating in G. We state the following slightly more general result for future reference (cf. [7]).

Lemma 2 *Let G be a CA graph and H be a hole of G. If $v \in V(G) \setminus V(H)$, then either v is adjacent to every vertex of H or $N_G(v) \cap V(H)$ induces a path in G.*

2.1 Balancedness of a superclass of Helly circular-arc graphs

A family $\mathcal{A}$ of nonempty sets has the Helly property if every nonempty subfamily of $\mathcal{A}$ consisting of pairwise intersecting sets has a nonempty intersection. A Helly circular-arc (HCA) graph [19] is a circular-arc graph admitting a circular-arc model $\mathcal{A}$ that satisfies the Helly property. We call $\mathcal{A}$ a *HCA model* of the graph. The class of HCA graphs contains all interval graphs because every set of intervals of a line has the Helly property [21]. Let G be a HCA graph and let $\mathcal{A}$ be a HCA model of G. Let us denote by A_v the

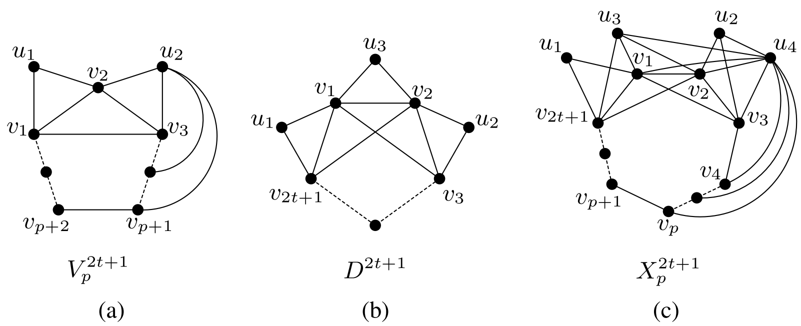

Fig. 5: Families of minimally non-balanced HCA graphs: (a) Family V_p^{2t+1}: The dotted paths joining v_3 and v_{p+1} resp. v_{p+2} and v_1 represent chordless even paths, not simultaneously empty. All vertices of the dotted path joining v_3 to v_{p+1} are adjacent to u_2. (b) Family D^{2t+1}: The dotted path joining v_3 and v_{2t+1} represents a nonempty even path of length $2t-2$. (c) Family X_p^{2t+1}: The dotted paths joining v_4 and v_p resp. v_{p+1} and v_{2t+1} represent any chordless even paths, both of them possibly empty, even simultaneously. The vertices of the dotted path joining v_4 to v_p are all adjacent to u_4.

arc of $\mathcal{A}$ that corresponds to the vertex $v \in V(G)$. For a clique Q of G, we call any point $p \in \bigcap_{v\in Q} A_v$ an *anchor* of Q. Since Q is inclusion-wise maximal, for each anchor p of Q and each $v \in V(G)$, it holds that $p \in A_v$ if and only if $v \in Q$. In [23], a linear-time recognition algorithm for HCA graphs was proposed, as well as a characterization by forbidden induced subgraphs of those CA graphs that are HCA graphs (see Theorem 5 on page 11).

In this subsection, we give a minimal forbidden induced subgraph characterization of balancedness for a superclass of HCA graphs. In order to do so, we introduce the graph families below, which are schematically represented in Figure 5.

- For each $t \geq 2$ and each p even such that $2 \leq p \leq 2t$, let V_p^{2t+1} be the graph with vertex set $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2\}$, such that $v_1v_2 \ldots v_{2t+1}v_1$ is a cycle whose only chord is v_1v_3, $N(u_1) = \{v_1, v_2\}$, and $N(u_2) = \{v_2, v_3, \ldots, v_{p+1}\}$.
- For each $t \geq 2$, let D^{2t+1} be the graph with $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3\}$ as vertex set such that $v_1v_2 \ldots v_{2t+1}v_1$ is a cycle whose only chords are $v_{2t+1}v_2$ and v_1v_3, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3\}$, and $N(u_3) = \{v_1, v_2\}$.
- For each $t \geq 2$ and each even p with $4 \leq p \leq 2t$, let X^{2t+1} be the graph with vertex set $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3, u_4\}$, $v_1v_2 \ldots v_{2t+1}v_1$ is a cycle whose only chords are $v_{2t+1}v_2$ and v_1v_3, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3, u_4\}$, $N(u_3) = \{v_{2t+1}, v_1, v_2, u_4\}$, and $N(u_4) = \{v_1, v_2, v_3, \ldots, v_p, u_2, u_3\}$.

In the three families of graphs above, $C = v_1v_2 \ldots v_{2t+1}v_1$ is an unbalanced cycle and consequently all their members are not balanced. In fact, we will see later that all these graphs are minimally non-balanced (see Corollary 4).

Our first result below is the minimal forbidden induced subgraph characterization of balanced graphs restricted to HCA graphs.

Theorem 3 *Let G be a HCA graph. Then, G is balanced if and only if G has no odd holes and contains no induced 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for any $t \geq 2$ and any valid p.*

Proof: The 'only if' part is clear because the class of balanced graphs is hereditary. Conversely, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally non-balanced; i.e., H is not balanced but each proper induced subgraph of H is balanced. Since G is a HCA graph, H also is so. The proof will be complete as soon as we prove that H is an odd hole, 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for some $t \geq 2$ and some valid p.

Since H is not balanced, the clique-matrix of H contains some square submatrix of odd order with two ones per row and per column. Clearly, this implies that the clique-matrix of H contains some square submatrix that is the clique-matrix of an odd chordless cycle. Therefore, there are some cliques $Q_1, Q_2, \ldots, Q_{2t+1}$ and vertices $v_1, v_2, \ldots, v_{2t+1}$ of H such that $\{v_1, v_2, \ldots, v_{2t+1}\} \cap Q_i = \{v_i, v_{i+1}\}$ for each $i \in \{1, 2, \ldots, 2t+1\}$ (all along the proof, subindices are to be understood modulo $2t+1$) for some $t \geq 1$. It is easy to verify that $C = v_1 v_2 \ldots v_{2t+1} v_1$ is an unbalanced cycle by setting $W_e := Q_i \setminus \{v_i, v_{i+1}\}$ for each edge $e = v_i v_{i+1}$ of C.

Suppose first that $t = 1$. Since C is an unbalanced cycle, there is some vertex $u_1 \in N_H(v_1 v_2) \setminus V(C)$ such that u_1 is not adjacent to v_3. Indeed, since $N_H(W_{v_1 v_2}) \cap N_H(v_1 v_2) \cap V(C) = \emptyset$ and $v_3 \in N_H(v_1 v_2) \cap V(C)$, there must be at least one vertex of $W_{v_1 v_2}$ that is not adjacent to v_3; clearly, it suffices to let u_1 be any such vertex. Similarly, there is some vertex $u_2 \in N_H(v_2 v_3) \setminus V(C)$ that is not adjacent to v_1, and some vertex $u_3 \in N_H(v_3 v_1) \setminus V(C)$ that is not adjacent to v_2. Now, $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ induces a pyramid in H. This implies that H itself is a pyramid because H is minimally non-balanced. So, if $t = 1$, then H equals a 3-sun, 1-pyramid or 2-pyramid (because the 3-pyramid is not a HCA graph).

From now on, we will assume, without loss of generality, that $t \geq 2$. Let $\mathcal{A}$ be a HCA model of H in a circle $\mathcal{C}$. Denote by A_i the arc of $\mathcal{A}$ corresponding to the vertex v_i for each $i \in \{1, 2, \ldots, 2t+1\}$. Fix an anchor p_j of the clique Q_j for each $j \in \{1, \ldots, 2t+1\}$. By construction, $p_j \in A_i$ if and only if $v_i \in Q_j$. Therefore, by hypothesis, $\{p_1, p_2 \ldots, p_{2t+1}\} \cap A_i = \{p_{i-1}, p_i\}$ for each $i \in \{1, \ldots, 2t+1\}$. Since $A_1, A_2, \ldots, A_{2t+1}$ are arcs of $\mathcal{C}$, there are only two possible orders for the anchors when traversing $\mathcal{C}$ in clockwise direction, either $p_1, p_2, \ldots, p_{2t+1}$ or $p_{2t+1}, \ldots, p_2, p_1$. So, we can assume, without loss of generality, that the anchors $p_1, p_2, \ldots, p_{2t+1}$ appear exactly in that order when traversing $\mathcal{C}$ in clockwise direction. Hence, $A_i \cap \{p_1, p_2, \ldots, p_{2t+1}\} = \{p_{i-1}, p_i\}$ implies that A_i is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_{i-2} and ends in p_{i+1} for each $i \in \{1, \ldots, 2t+1\}$. We now prove the following three claims about C.

Claim 1 *All chords of C are short.*

If $t = 2$, all possible chords of C are short. So, suppose that $t \geq 3$. Since A_i is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_{i-2} and ends in p_{i+1} for each $i \in \{1, \ldots, 2t+1\}$, it follows that if the arc A_i intersects A_j for some $j \in \{1, \ldots, 2t+1\}$ then $i = j-2, j-1, j, j+1$, or $j+2$ (modulo $2t+1$). We conclude that each chord of C is short, as claimed.

Claim 2 *Any set of three vertices of C that induces a triangle in H consists of three consecutive vertices of C.*

Suppose, for the purpose of contradiction, that there is some set S of three vertices of C that induces a triangle T in H but, nevertheless, S does not consist of three consecutive vertices of C. Notice that if each vertex of S were consecutive in C to some other vertex of S, then S would consist of three consecutive vertices of C. So, necessarily, there must be some vertex s_1 of S such that s_1 is not consecutive in C to any vertex of $S \setminus \{s_1\}$. By symmetry, we can assume that $s_1 = v_1$ and, since all chords of C are short, $S = \{v_1, v_3, v_{2t}\}$. As C is odd and each of its chords is short, necessarily $t = 2$. Consequently, $S = \{v_1, v_3, v_4\}$ is contained in some clique of H, that should have some anchor q. Nevertheless, since A_1 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_4 and ends in p_2, A_3 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_1 and ends in p_4, and A_4 is contained in the clockwise open arc of $\mathcal{C}$ that starts in p_2 and ends in p_5, there is no suitable position in $\mathcal{C}$ for q. This contradiction proves that indeed any set of three vertices of C that induces a triangle in H consists of three consecutive vertices of C, as claimed.

Claim 3 *Every two chords of C are crossing.*

Suppose, for the purpose of contradiction, that C has two different chords $e_i = v_{i-1}v_{i+1}$ and $e_j = v_{j-1}v_{j+1}$ that are not crossing (recall that all chords are short by Claim 1). Notice that it is possible that e_i and e_j share one endpoint. We will show that $H - \{v_i, v_j\}$ is not balanced. Indeed, consider the cycle $C' = v_1v_2 \ldots v_{i-1}v_{i+1} \ldots v_{j-1}v_{j+1} \ldots v_{2t+1}v_1$. For each edge e of C', define $W'_e = \emptyset$, if $e = e_i$ or e_j; and $W'_e = W_e$, otherwise. Since all the triangles of C are induced by three consecutive vertices of C by Claim 2, C' and the W'_e's satisfy the definition of an unbalanced cycle. Indeed, for each edge e of C', either $W'_e = W_e$ and $N(W'_e) \cap N(e) \cap V(C') \subseteq N(W_e) \cap N(e) \cap V(C) = \emptyset$, or $e = e_k$ for $k \in \{i, j\}$ and $N(W_e) \cap N(e) \cap V(C') \subseteq N(e) \cap (V(C) \setminus \{v_k\}) = \emptyset$ because, by Claim 2, the only vertex of C with which vertices v_{k-1} and v_{k+1} can form a triangle in H is v_k. Therefore, $H - \{v_i, v_j\}$ is not balanced, a contradiction with the minimality of H. This contradiction shows that indeed every two chords of C are crossing, as claimed.

With the help of the three previous claims, we complete the proof of Theorem 3. Notice that if C has no chords, then, by the minimality of H, $H = C_{2t+1}$, as required. Therefore, we will assume that C contains at least one chord. Since all chords of C are short and crossing by Claims 1 and 3, either C has exactly one chord that is short or C has exactly two chords that are short and are crossing. We divide the remaining proof into two cases corresponding to the former and the latter cases.

Case 1 *C has exactly one chord that is short.*

Without loss of generality, let v_1v_3 be the only chord of C. Since C is an unbalanced cycle, there exists $u_1 \in N_H(v_1v_2) \setminus V(C)$ such that u_1 is not adjacent to v_3. Analogously, there exists $u_2 \in N_H(v_2v_3) \setminus V(C)$ such that u_2 is not adjacent to v_1. By minimality, $V(H) = V(C) \cup \{u_1, u_2\}$. Let $p = |N_H(u_2) \cap V(C)|$ and $q = |N_H(u_1) \cap V(C)|$. By construction, $2 \leq p, q \leq 2t$. By Lemma 2 applied to the hole induced by $V(C) \setminus \{v_2\}$, $N_H(u_2) \cap V(C) = \{v_2, v_3, v_4, \ldots, v_{p+1}\}$ and, by symmetry, $N_H(u_1) \cap V(C) = \{v_2, v_1, v_{2t+1}, v_{2t}, \ldots, v_{2t-q+4}\}$ (where for $q = 2$, we mean that $N_H(u_1) \cap V(C) = \{v_2, v_1\}$).

Suppose, for the purpose of contradiction, that u_1 is adjacent to u_2. If u_2 were adjacent to v_{2t+1}, then either $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 2-pyramid in H or $\{v_{2t+1}, v_1, v_3, u_1, u_2\}$ would induce a $K_{2,3}$ in H, depending on whether u_1 is adjacent to v_{2t+1} or not, respectively. Since H is a minimally non-balanced CA graph and $K_{2,3}$ is not a CA graph, we conclude that u_2 is not adjacent to v_{2t+1}. If u_1 were adjacent to v_{2t+1}, then $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 1-pyramid in H. This contradiction shows that u_1 is not adjacent to v_{2t+1}, and this means that $q = 2$. Symmetrically, $p = 2$. But then, either $t = 2$ and $\{v_1, v_3, v_4, v_5, u_1, u_2\}$ induces a domino in H, or $t \geq 3$ and

$\{v_1, v_3, u_2, u_1, v_5\}$ induces a $C_4 \cup K_1$ in H, which are not CA graphs, a contradiction. This contradiction arose from assuming that u_1 and u_2 were adjacent, so we conclude that u_1 is not adjacent to u_2.

If p were odd, then $u_2 v_{p+1} v_{p+2} \dots v_{2t+1} v_1 v_2 u_2$ would be an odd hole in H, contradicting the minimality of H. Thus, p is even and, by symmetry, q is also even. If $t = 2$, then, up to symmetry, either $p = q = 4$ and $H = \overline{C_7}$, or $q = 2$ and $H = V_p^5$ for some $p \in \{2, 4\}$, as desired. So, without loss of generality, assume that $t \geq 3$. If $N_H(u_1) \cap N_H(u_2) \neq \{v_2\}$, then, since p and q are even, there would exist some k such that $5 \leq k \leq 2t$ and $v_k \in N_H(u_1) \cap N_H(u_2)$; but then, $\{v_1, u_1, v_k, u_2, v_3\}$ would induce a C_5 in H, in contradiction with the minimality of H. This contradiction shows that $N_H(u_1) \cap N_H(u_2) = \{v_2\}$. If $p \neq 2$ and $q \neq 2$, then $u_2 v_{p+1} v_{p+2} \dots v_{2t-q+4} u_1 v_2 u_2$ would be an odd hole in H, contradicting the minimality of H. Therefore, we can assume that $q = 2$, and finally $H = V_p^{2t+1}$ for some p even such that $2 \leq p \leq 2t$.

Case 2 *C has exactly two chords that are short and are crossing.*

Since the two chords are crossing, we assume, without loss of generality, that the chords of C are $v_1 v_3$ and $v_{2t+1} v_2$. Since C is an unbalanced cycle, there is some $u_1 \in N_H(v_{2t+1} v_1) \setminus V(C)$ such that u_1 is not adjacent to v_2 and there is some $u_2 \in N_H(v_2 v_3) \setminus V(C)$ such that u_2 is not adjacent to v_1.

Let $r = |N_H(u_2) \cap V(C)|$. By construction, $2 \leq r \leq 2t$ and, by Lemma 2 applied to the hole induced by $V(C) \setminus \{v_2\}$, $N_H(u_2) \cap V(C) = \{v_2, v_3, v_4, \dots, v_{r+1}\}$. If $r = 2t$, then $\{v_{2t+1}, v_1, v_2, v_3, u_1, u_2\}$ would induce a proper 1-, 2- or 3-pyramid in H (depending on the existence or not of the edges $u_1 u_2$ and $u_1 v_3$), a contradiction with the minimality of H. If r is even and $2 < r < 2t$, then the cycle $u_2 v_{r+1} v_{r+2} \dots v_{2t} v_{2t+1} v_2 u_2$ would be a proper odd hole in H, a contradiction. If r were odd and $r \neq 3$, then the cycle $u_2 v_{r+1} v_{r+2} \dots v_{2t} v_{2t+1} v_1 v_3 u_2$ would be a proper odd hole in H, a contradiction. So, $r = 2$ or 3. Symmetrically, if $s = |N_H(u_1) \cap V(C)|$, then $s = 2$ or 3 and, by Lemma 2 applied to the hole induced by $V(C) \setminus \{v_1\}$, $N_H(u_1) \cap V(C) = \{v_{2t+1}, v_1\}$ or $\{v_{2t}, v_{2t+1}, v_1\}$, respectively.

Suppose, for the purpose of contradiction, that u_1 and u_2 are adjacent. Then, the set $\{u_1, v_1, v_2, u_2\}$ induces a C_4 in H, which must be dominating because H is a CA graph. If $t = 2$, then at least one of u_1 and u_2 should be adjacent to v_4 and either $\{u_1, u_2, v_1, v_3, v_4\}$ or $\{u_1, u_2, v_2, v_4, v_5\}$ would induce a $K_{2,3}$ or $V(C) \cup \{u_1, u_2\}$ would induce a proper $\overline{C_7}$ in H. (Notice that indeed $V(C) \cup \{u_1, u_2\}$ induces a proper subgraph of H because, by the definition of an unbalanced cycle, $W_{v_1 v_2} \subseteq V(H) \setminus V(C)$ and $N_H(W_{v_1 v_2}) \cap \{v_3, v_4\} = \emptyset$, which implies $W_{v_1 v_2} \neq \emptyset$ and, by construction, $W_{v_1 v_2} \cap (V(C) \cup \{u_1, u_2\}) = \emptyset$.) If $t \geq 3$, then $\{u_1, u_2, v_1, v_2, v_5\}$ would induce $C_4 \cup K_1$ in H. So, in all cases we reach a contradiction with the minimality of H. These contradictions prove that u_1 and u_2 are nonadjacent.

We claim that $r = s = 2$. Indeed, if $r = s = 3$, then $v_1 v_2 u_2 v_4 v_5 \dots v_{2t} u_1 v_1$ would be an odd hole in H, a contradiction. Alternatively, if $r = 3$ and $s = 2$, then $C' = v_1 v_2 u_2 v_4 v_5 \dots v_{2t+1} v_1$ would be a cycle whose only chord is $v_{2t+1} v_2$, $N_H(u_1) \cap V(C') = \{v_{2t+1}, v_1\}$, $N_H(v_3) \cap V(C') = \{v_1, v_2, u_2, v_4\}$ and, therefore, $V(C) \cup \{u_1, u_2\}$ would induce a proper V_4^{2t+1} in H, a contradiction. (Recall that $V(C) \cup \{u_1, u_2\} \neq V(H)$ from the discussion in the paragraph above.) The case $r = 2$ and $s = 3$ is symmetric. We conclude that our claim, $r = s = 2$, is true; in other words, $N_H(u_1) \cap V(C) = \{v_{2t+1}, v_1\}$ and $N_H(u_2) \cap V(C) = \{v_2, v_3\}$.

Suppose that

$$\text{there is some } u_3 \in N_H(v_1 v_2) \setminus V(C) \text{ such that } u_3 v_{2t+1}, u_3 v_3 \notin E(H). \qquad (*)$$

Then, by minimality, $V(H) = V(C) \cup \{u_1, u_2, u_3\}$. By Lemma 2 applied to the hole induced by $V(C) \setminus \{v_2\}$, $N_H(u_3) \cap V(C) = \{v_1, v_2\}$. If u_1 were adjacent to u_3, then either $t = 2$ and $\{v_2, v_3, v_4, v_5, u_1, u_3\}$

would induce a domino in H, or $t \geq 3$ and $\{u_1, v_{2t+1}, v_2, u_3, v_5\}$ would induce $C_4 \cup K_1$ in H, which are not CA graphs, a contradiction. So, u_1 is nonadjacent to u_3 and, symmetrically, u_2 is nonadjacent to u_3. We conclude that, if (*) holds, $H = D^{2t+1}$, as desired.

It only remains to consider the case when (*) does not hold. Since C is an unbalanced cycle, this means that there are two adjacent vertices u_3 and u_4 such that $u_3, u_4 \in N_H(v_1v_2) \setminus V(C)$, u_3 is adjacent to v_{2t+1} but not to v_3, and u_4 is adjacent to v_3 but not to v_{2t+1}.

We notice that $N_H(u_3) \cap N_H(u_4) \cap V(C) \neq \{v_1, v_2\}$, since otherwise there would be some k such that $4 \leq k \leq 2t$ and $v_k \in N_H(u_3) \cap N_H(u_4)$ and $\{v_{2t+1}, v_1, v_3, v_k, u_3, u_4\}$ would induce a proper 0-, 1-, or 2-pyramid in H depending on whether the number of inequalities holding with equality in $4 \leq k \leq 2t$ were 0, 1, or 2, respectively, contradicting the minimality of H.

Let $p = |N_H(u_4) \cap V(C)|$ and $q = |N_H(u_3) \cap V(C)|$. By construction, $3 \leq p, q \leq 2t$. By Lemma 2 applied to the hole induced by $V(C) \setminus \{v_2\}$, $N_H(u_4) \cap V(C) = \{v_1, v_2, v_3, \ldots, v_p\}$ and $N_H(u_3) \cap V(C) = \{v_2, v_1, v_{2t+1}, \ldots, v_{2t-q+4}\}$. If p were odd and $p \neq 3$, then $v_1u_4v_pv_{p+1}\ldots v_{2t+1}v_1$ would be a proper odd hole in H, a contradiction. So, $p = 3$ or p is even. Symmetrically, $q = 3$ or q is even. If p and q had the same parity, then $u_3u_4v_pv_{p+1}\ldots v_{2t-q+4}u_3$ would be a proper odd hole of H (recall that $N_H(u_3) \cap N_H(u_4) \cap V(C) = \{v_1, v_2\}$), a contradiction. By symmetry, we will assume, without loss of generality, that p is even, $p \geq 4$, and $q = 3$. In particular, u_4 is adjacent to v_4.

Notice that u_2 is not adjacent to u_3, since otherwise $u_2v_3v_4\ldots v_{2t+1}u_3u_2$ would be a proper odd hole of H. Moreover, u_2 is adjacent to u_4, since otherwise $\{v_2, v_3, v_4, u_3, u_4, u_2\}$ would induce a proper 3-sun in H. So, $N_H(u_2) = \{v_2, v_3, u_4\}$. (Recall that we already proved that u_1 and u_2 are nonadjacent.)

If u_1 were adjacent to u_4, then $\{v_{2t+1}, v_1, v_2, u_1, u_2, u_4\}$ would induce a proper 1-pyramid in H, contradicting the minimality of H. So, u_1 is nonadjacent to u_4. Finally, if u_1 were adjacent to u_3, then $C' = u_3v_2v_3\ldots v_{2t+1}u_3$ would be a cycle whose only chord is $v_{2t+1}v_2$, $N_H(u_1) \cap V(C') = \{v_{2t+1}, u_3\}$, $N_H(u_4) \cap V(C') = \{u_3, v_2, v_3, \ldots, v_p\}$ and, therefore, since u_1 and u_4 are nonadjacent, $V(C') \cup \{u_1, u_4\}$ would induce a proper V_p^{2t+1} in H, a contradiction. This contradiction shows that u_1 is nonadjacent to u_3 and we conclude that $N_H(u_1) = \{v_{2t+1}, v_1\}$. We proved that $H = X_p^{2t+1}$ where p is even and $4 \leq p \leq 2t$, as required. □

It is easy to see that among the forbidden induced subgraphs that characterize balancedness in Theorem 3 there are no two of them such that one is a proper induced subgraph of the other. Therefore, Theorem 3 is indeed a characterization by *minimal* forbidden induced subgraphs. In particular, we obtain the following result.

Corollary 4 *The graphs V_p^{2t+1}, D^{2t+1}, and X_p^{2t+1} are minimally non-balanced for any $t \geq 2$ and any valid p.*

We will extend Theorem 3 to a superclass of HCA graphs; namely, the class of $\{$net,U_4,$S_4\}$-free CA graphs (cf. Figure 3). This extension will also serve as a basis for the characterizations in the following two subsections.

For that, let us firstly present the forbidden induced subgraph characterization of those CA graphs that are HCA graphs given in [23]. Let an *obstacle* be a graph H containing a clique $Q = \{v_1, v_2, \ldots, v_t\}$ where $t \geq 3$ and such that for each $i \in \{1, \ldots, t\}$, at least one of the following assertions holds (where in both assertions, v_{t+1} means v_1):

(Y_1) $N(w_i) \cap Q = Q \setminus \{v_i, v_{i+1}\}$, for some $w_i \in V(H) \setminus Q$.

(Y_2) $N(u_i) \cap Q = Q \setminus \{v_i\}$ and $N(z_i) \cap Q = Q \setminus \{v_{i+1}\}$, for some adjacent vertices $u_i, z_i \in V(H) \setminus Q$.

With this definition, the characterization of those CA graphs that are HCA graphs runs as follows.

Theorem 5 ([23]) *Let G be a CA graph. Then, G is a HCA graph if and only if G contains no induced obstacle.*

Notice that obstacles are not necessarily *minimal*; i.e., there are obstacles that contain proper induced obstacles. For instance, $\overline{2C_5}$ is an obstacle and contains a proper induced $\overline{2P_4}$, which is also an obstacle. In addition, there are minimal obstacles that are not CA graphs; e.g., antenna and $\overline{C_6}$ are minimal obstacles that are not CA graphs. Our next result determines all the $\{$1-pyramid,2-pyramid$\}$-free minimal obstacles that are CA graphs. Recall that for each $t \geq 3$, S_t denotes the complete t-sun.

Theorem 6 *Let H be a $\{$1-pyramid,2-pyramid$\}$-free minimal obstacle which is a CA graph. Then, H is the 3-pyramid, U_4, or $\overline{S_t}$ for some $t \geq 3$.*

Proof: Let $Q = \{v_1, \ldots, v_t\}$, the w_i's, the u_i's, and the z_i's as in the definition of an obstacle. All along the proof, subindices should be understood modulo t.

Let us consider first the case where $t = 3$. Suppose that (Y_2) holds for at least two values of i, say $i = 1$ and $i = 2$. Then, $\{u_1, z_1, z_2\}$ is a complete and $\{v_1, v_2, v_3, u_1, z_1, z_2\}$ induces a 3-pyramid, since otherwise $\{v_1, v_2, v_3, u_1, z_1, z_2\}$ would induce a 1-pyramid or a 2-pyramid. Hence, by minimality, $H =$ 3-pyramid. Consider now the case where (Y_2) holds for exactly one value of i, say $i = 1$, and, consequently, (Y_1) holds for $i = 2$ and $i = 3$. We claim that $\{u_1, z_1\}$ is anticomplete to w_2. Indeed, if w_2 were adjacent to z_1, then $\{v_1, v_2, v_3, w_2, z_1, u_1\}$ would induce a 1-pyramid or a 2-pyramid in H, a contradiction. In addition, if w_2 were adjacent to u_1, then $\{v_1, v_2, u_1, z_1, w_2\}$ would induce a $K_{2,3}$ in G, which is not a CA graph. We proved that $\{u_1, z_1\}$ is anticomplete to w_2 and, symmetrically, to w_3. Also notice that w_2 and w_3 are nonadjacent, since otherwise $\{v_1, v_2, w_2, w_3, u_1, z_1\}$ would induce a domino, which is not a CA graph. Then, by minimality, $H = U_4$, as desired. Finally, assume that (Y_1) holds for each $i \in \{1, 2, 3\}$. Necessarily $\{w_1, w_2, w_3\}$ is a stable set, since otherwise G would contain an induced $C_4 \cup K_1$, G_3 (see Figure 4), or $\overline{C_6}$ which are not CA graphs. By minimality, $H =$ net $= \overline{S_3}$, as desired.

From now on, we assume that $t \geq 4$. Suppose, by way of contradiction, that (Y_2) holds for some i, say $i = 1$. On the one hand, if (Y_1) held for $i = 3$, then $\{v_1, v_2, v_3, u_1, z_1, w_3\}$ would induce a 1-pyramid, 2-pyramid, or a proper 3-pyramid in H, a contradiction. On the other hand, if (Y_2) held for $i = 3$, then $\{v_1, v_2, v_3, u_1, z_1, u_3\}$ would induce a 1-pyramid, 2-pyramid or a proper 3-pyramid in H, a contradiction. These contradictions arose from assuming that (Y_2) held for some i. We conclude that, if $t \geq 4$, then (Y_2) does not hold for any $i \in \{1, \ldots, t\}$ and, by definition of obstacle, (Y_1) holds for each $i \in \{1, \ldots, t\}$. By minimality, the vertices of H are $Q \cup W$ where $W = \{w_1, w_2, \ldots, w_t\}$. We claim that W is a stable set and, consequently, $H = \overline{S_t}$. We divide the proof of the claim into two cases: $t = 4$ and $t \geq 5$.

Assume that $t = 4$. Suppose, for the purpose of contradiction, that W is not a stable set. Suppose first that w_i is adjacent to w_{i+1} for some i, say w_3 is adjacent to w_4. Necessarily w_1 is nonadjacent to w_4, since otherwise $\{v_1, v_2, v_3, w_1, w_3, w_4\}$ would induce a 1-pyramid or a 2-pyramid in H (depending on the adjacency between w_1 and w_3), a contradiction. In addition, w_1 is nonadjacent to w_3, since otherwise $\{w_1, v_1, w_4, v_3, w_3\}$ would induce a $K_{2,3}$, which is not a CA graph. Symmetrically, w_2 is nonadjacent to w_3 and w_4. On the one hand, if w_1 and w_2 are adjacent, then $\{w_2, v_1, w_3, w_4, v_3, w_1\}$ induces a domino in G, which is not a CA graph. On the other hand, if w_1 and w_2 are nonadjacent, then $\{v_1, v_2, v_3, w_1, w_2, w_3, w_4\}$ induces a proper U_4 in H, a contradiction with the minimality of H.

These contradictions prove that w_i is not adjacent to w_{i+1} for any i. Notice that also w_i and w_{i+2} are nonadjacent, since otherwise $\{v_i, w_i, w_{i+2}, v_{i+3}, w_{i+3}\}$ would induce $C_4 \cup K_1$ in G, which is not a CA graph. We conclude that W is a stable set and $H = \overline{S_4}$, as claimed.

It only remains to consider the case where $t \geq 5$. Let S be any unordered pair of vertices from W. Since $t \geq 5$, S can be extended to a set $S' = \{w_i, w_j, w_{j+1}\}$ of three vertices where i and j are not consecutive modulo t and neither are i and $j+1$. Notice that S' is a stable set in H, since otherwise $\{v_i, v_j, v_{j+2}, w_i, w_j, w_{j+1}\}$ would induce a 1-pyramid, a 2-pyramid, or a proper 3-pyramid in H, a contradiction. Since S' is a stable set, so is S. Since S is any pair of vertices from W, W is a stable set and $H = \overline{S_t}$, as claimed.

Finally, notice that 3-pyramid, U_4 and $\overline{S_t}$ for $t \geq 3$ are obstacles, are CA graphs, and none of them is a proper induced subgraph of any of the others. □

As a corollary of Theorems 5 and 6, we obtain a minimal forbidden induced subgraph characterization of HCA graphs within the class of {1-pyramid,2-pyramid}-free CA graphs.

Corollary 7 *Let G be a {1-pyramid,2-pyramid}-free CA graph. Then, G is a HCA graph if and only if it contains no induced 3-pyramid, U_4, or $\overline{S_t}$ for any $t \geq 3$.*

Since net, U_4, and S_4 are obstacles, the class of {net,U_4,S_4}-free CA graphs is indeed a superclass of HCA graphs. We now prove the main result of this subsection, which is an extension of the characterization of Theorem 3 to the class of {net,U_4,S_4}-free CA graphs.

Corollary 8 *Let G be a {net,U_4,S_4}-free CA graph. Then, G is balanced if and only if G has no odd holes and contains no induced pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for any $t \geq 2$ and any valid p.*

Proof: If G is a HCA graph, the result reduces to Theorem 3. So, assume that G is not a HCA graph. Then, by Corollary 7 and since G is {net,U_4,S_4}-free, G contains an induced 1-pyramid, 2-pyramid, or 3-pyramid or an induced $\overline{S_t}$ for some $t \geq 5$ (notice that $\overline{S_3}$ = net and $\overline{S_4} = S_4$). Since $\overline{S_t}$ contains an induced 3-sun for every $t \geq 5$, we conclude that G is not balanced and contains an induced pyramid. □

Corollary 8 is crucial in the proof of the main results in the next two subsections.

2.2 Balancedness of claw-free circular-arc graphs

In this subsection we will characterize, by minimal forbidden induced subgraphs, those claw-free CA graphs that are balanced. A *proper circular-arc graph* (PCA) is a CA graph admitting a CA model in which no arc properly contains another. The class of claw-free CA graphs is a superclass of the class of PCA graphs, as follows from the forbidden induced subgraph characterization of PCA graphs in [31].

By Corollary 8, in order to characterize balanced graphs within claw-free CA graphs, it will be enough to study the balancedness of those claw-free CA graphs containing an induced net (because claw-free graphs contain neither induced U_4's nor induced S_4's). The following lemma will be of help in analyzing the structure of claw-free CA graphs containing an induced net.

Lemma 9 ([7]) *Let G be a claw-free CA graph containing a net induced by $H = \{t_1, t_2, t_3, s_1, s_2, s_3\}$, where $\{t_1, t_2, t_3\}$ induces a triangle and s_i is adjacent to t_i for each $i \in \{1, 2, 3\}$. If v is a vertex of $G - H$, then $N_G(v) \cap H$ is either $\{s_i, t_i\}$, or $\{t_1, t_2, t_3, s_i\}$, or $\{s_i, t_i, t_{i+1}, s_{i+1}\}$, for some $i \in \{1, 2, 3\}$ (where subindices should be understood modulo 3).*

A graph G arises from a graph H by *vertex replication* if G can be obtained from H by replacing each vertex x of H by a nonempty complete graph M_x and adding all possible edges between M_x and M_y if and only if x and y are adjacent in H. In [7], a slightly stronger variant of the above lemma is used to study the structure of chordal claw-free CA graphs containing an induced net. The proof in [7] can be easily adapted to prove the following related result in which chordality is not required. For the sake of completeness, we give the adapted proof.

Theorem 10 ([7]) *If G is a claw-free CA graph containing an induced net and containing no induced 3-sun, then G arises from the net by vertex replication.*

Proof: The proof will be by induction on the number of vertices of G. If $|V(G)| = 6$, then G equals a net, which trivially arises from the net by vertex replication. So, assume that $|V(G)| > 6$. Then, there is some vertex v of G such that $G - \{v\}$ contains an induced net. Since $G - \{v\}$ is also a claw-free graph containing an induced net and containing no induced 3-sun, by induction hypothesis, $G - \{v\}$ arises from a net by vertex replication; i.e., the vertices of $V(G - \{v\})$ can be partitioned into nonempty completes S_1, S_2, S_3, T_1, T_2, T_3 such that T_1, T_2, T_3 are mutually complete and T_i is complete to S_i and anticomplete to S_{i+1} and S_{i+2}, for each $i \in \{1, 2, 3\}$ (where subindices along the proof should be understood modulo 3). Let $s_1 \in S_1$, $s_2 \in S_2$, $s_3 \in S_3$, $t_1 \in T_1$, $t_2 \in T_2$, and $t_3 \in T_3$. Let H be the net induced by $\{s_1, s_2, s_3, t_1, t_2, t_3\}$ in $G - \{v\}$. By Lemma 9, $N_G(v) \cap H = \{s_i, t_i\}$ or $N_G(v) \cap H = \{t_1, t_2, t_3, s_i\}$ for some $i \in \{1, 2, 3\}$. (Notice that the fact that G contains no induced 3-sun prevents $N_G(v) \cap H = \{t_i, s_i, t_{i+1}, s_{i+1}\}$ from holding.)

Suppose first that $N_G(v) \cap H = \{t_i, s_i\}$ for some $i \in \{1, 2, 3\}$. Let $j \in \{1, 2, 3\}$. If $s'_j \in S_j$, then, applying Lemma 9 to the net induced by $\{t_1, t_2, t_3, s'_j, s_{j+1}, s_{j+2}\}$, it follows that v is adjacent to s'_j if and only if $j = i$. Thus, v is complete to S_i and anticomplete to S_{i+1} and S_{i+2}. Using the same strategy, we can prove that v is complete to T_i and anticomplete to T_{i+1} and T_{i+2}. Thus, we can obtain a partition of the vertices of G showing that G is arises from a net by vertex replication (replacing S_i by $S_i \cup \{v\}$).

Finally, consider that $N_G(v) \cap H = \{t_1, t_2, t_3, s_i\}$. Reasoning as in the above paragraph, it follows that v is complete to T_1, T_2, T_3, and S_i, and v is anticomplete to S_{i+1} and S_{i+2}. Thus, we obtain a partition of the vertices of G showing that G arises from a net by vertex replication (replacing T_i by $T_i \cup \{v\}$). $\square$

Recall that a graph is minimally non-balanced if it is not balanced but each of its proper induced subgraphs is balanced. It is clear that minimally non-balanced graphs have no true twins, since the removal of one of the true twins only eliminates a repeated column in the clique-matrix. Now we state and prove the main result of this subsection.

Theorem 11 *Let G be a claw-free CA graph. Then, G is balanced if and only if G has no odd holes and contains no induced pyramids and no induced $\overline{C_7}$.*

Proof: The 'only if' part is clear. In order to prove the 'if' part, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally non-balanced. Since G is a claw-free CA graph, H also is so. The proof will be complete if we prove that H is an odd hole, a pyramid, or $\overline{C_7}$. Suppose, for the purpose of contradiction, that H is not net-free. By Theorem 10, H is a net, has true twins, or contains an induced 3-sun. Since the net is balanced and since minimally non-balanced graphs have no true twins, G contains an induced 3-sun. By minimality, H is a 3-sun, a contradiction with the fact that H is not net-free. This contradiction proves that H is net-free. Since U_4 and S_4 are not claw-free, H is $\{$net,U_4,$S_4\}$-free and Corollary 8 implies that H has an odd hole or contains an induced pyramid or $\overline{C_7}$

(because each of V_p^{2t+1}, D^{2t+1}, and X_p^{2t+1} contains an induced claw for each $t \geq 2$ and each valid p). By the minimality of H, we conclude that H is an odd hole, a pyramid, or $\overline{C_7}$, as required. □

As PCA graphs are claw-free, and the odd holes, the pyramids, and $\overline{C_7}$ are all PCA graphs, the minimal forbidden induced subgraphs for balancedness within PCA graphs are the same as those within claw-free CA graphs.

2.3 Balancedness of gem-free circular-arc graphs

In this subsection we will give minimal forbidden induced subgraph characterizations of balanced gem-free CA graphs.

Lemma 12 *Let G be a gem-free CA graph that contains an induced net or an induced U_4. Then, G either has true twins or has a cutpoint.*

Proof: Assume that G has no true twins. We will prove that G has a cutpoint.

Consider first the case where G contains an induced U_4. That is, there is some chordless cycle $C = u_1u_2u_3u_4u_1$ in G, some vertex z that is complete to $V(C)$, and a pair of nonadjacent vertices p_1, p_2 of G such that $N_G(p_i) \cap (V(C) \cup \{z\}) = \{u_i\}$ for each $i \in \{1, 2\}$. Since G is a CA graph, $V(C)$ is a dominating set of G. Let v be a vertex of G not in $V(C) \cup \{p_1, p_2, z\}$. We will analyze the possibilities for the nonempty set $N_G(v) \cap V(C)$.

Suppose, for the purpose of contradiction, that v has two neighbors on C. Then, they are consecutive vertices of C by Lemma 2. So, $N_G(v) \cap V(C) = \{u_i, u_{i+1}\}$ for some $i \in \{1, 2, 3, 4\}$ (from now on, subindices should be understood modulo 4). If v were not adjacent to z, then $\{v, u_i, z, u_{i+2}, u_{i+1}\}$ would induce a gem in G. If v were adjacent to z, then $\{v, u_{i+1}, u_{i+2}, u_{i+3}, z\}$ would induce a gem in G. Since G is gem-free, we conclude that $|N_G(v) \cap V(C)| \neq 2$.

Now, for each $i \in \{1, 2, 3, 4\}$, let V_i be the set of vertices not in $V(C)$ whose only neighbor in C is u_i. In particular, $p_i \in V_i$ for each $i \in \{1, 2\}$. Let Z be the set of vertices not in $V(C)$ that are complete to $V(C)$, so $z \in Z$. Finally, for each $i \in \{1, 2, 3, 4\}$, let $\bar{V}_i$ be the set of vertices not in $V(C)$ whose only non-neighbor in C is u_i. Notice that every $v \in V(G) \setminus V(C)$ belongs to one of the sets in $\{V_i, \bar{V}_i\}_{i \in \{1,2,3,4\}}$ or Z.

Claim 1 *V_i is anticomplete to V_j for every $i \neq j$.*

Indeed, if $v_i \in V_i$ and $v_j \in V_j$ were adjacent, then $V(C) \cup \{v_i, v_j\}$ would induce either a domino or the graph G_2 in Figure 4, which are not CA graphs, a contradiction.

Claim 2 *V_i is anticomplete to Z for every $i \in \{1, 2, 3, 4\}$.*

Indeed, if $v_i \in V_i$ and $w \in Z$ were adjacent, then $\{v_i, u_i, u_{i+1}, u_{i+2}, w\}$ would induce a gem in G, a contradiction.

Claim 3 *Z is a complete.*

Indeed, if w, w' in Z were nonadjacent, then, by the previous claim, both of them would be nonadjacent to p_2 and $\{u_1, w, u_3, w', p_2\}$ would induce $C_4 \cup K_1$ in G, which is not a CA graph, a contradiction.

Claim 4 *$\bar{V}_i$ is a complete and is complete to Z for every $i \in \{1, 2, 3, 4\}$.*

Indeed, if $\bar{v}_i, \bar{v}'_i$ in $\bar{V}_i$ were nonadjacent, then $\{\bar{v}_i, \bar{v}'_i, u_i, u_{i-1}, u_{i+1}\}$ would induce $K_{2,3}$ in G, which is not a CA graph, a contradiction. And, if $\bar{v}_i \in \bar{V}_i$ and $w \in Z$ were nonadjacent, then $\{\bar{v}_i, u_{i+2}, w, u_i, u_{i+1}\}$ would induce a gem in G, also a contradiction.

By the previous claims, all the vertices in Z are true twins. So, since G has no true twins, we conclude that $Z = \{z\}$.

Claim 5 *$\bar{V}_i$ is complete to $\bar{V}_{i+1}$ and anticomplete to $\bar{V}_{i+2}$ for every $i \in \{1, 2, 3, 4\}$.*

Let $\bar{v}_i \in \bar{V}_i$ and $\bar{v}_{i+1} \in \bar{V}_{i+1}$. If $\bar{v}_i$ and $\bar{v}_{i+1}$ were nonadjacent, then $\{u_i, \bar{v}_{i+1}, u_{i+2}, \bar{v}_i, u_{i+3}\}$ would induce a gem in G, a contradiction. If $\bar{v}_i$ were adjacent to some $\bar{v}_{i+2} \in \bar{V}_{i+2}$, then $\{u_i, \bar{v}_{i+2}, \bar{v}_i, u_{i+2}, u_{i+1}\}$ would induce a gem in G, a contradiction.

Claim 6 *$\bar{V}_i$ is anticomplete to V_j for every $j \neq i + 2$.*

Let $\bar{v}_i \in \bar{V}_i$ and $v_j \in V_j$ and suppose, by way of contradiction, they are adjacent. If $j = i$, then $\{u_i, \bar{v}_i, u_{i+1}, u_{i+3}, v_j\}$ induces a $K_{2,3}$ in G, that is not a CA graph, a contradiction. If $j = i \pm 1$, then $\{v_j, u_{i+1}, u_{i+2}, u_{i+3}, \bar{v}_i\}$ induces a gem in G, also a contradiction. These contradictions prove that $\bar{v}_i$ and v_j are nonadjacent unless $j = i + 2$.

Claim 7 *$\bar{V}_i$ is empty for every $i \in \{1, 2, 3, 4\}$.*

Suppose, by way of contradiction, that $\bar{V}_i$ is nonempty for some $i \in \{1, 2, 3, 4\}$ and let $\bar{v}_i \in \bar{V}_i$. Since $\bar{v}_i$ is not a true twin of v_{i+2}, by the previous claims, there must be a vertex v_{i+2} in V_{i+2} that is nonadjacent to $\bar{v}_i$. But then, $\{\bar{v}_i, u_{i+3}, u_i, u_{i+1}, v_{i+2}\}$ induces a $C_4 \cup K_1$ in G, that is not a CA graph, a contradiction.

By all the previous claims, u_1 and u_2 are cutpoints of G, as required. This completes the proof when G contains an induced U_4.

It only remains to consider the case where G contains no induced U_4 but a net induced by $H = T \cup S$ where $T = \{t_1, t_2, t_3\}$ is a complete, $S = \{s_1, s_2, s_3\}$ is a stable set and $N_G(s_i) \cap T = \{t_i\}$ for each $i \in \{1, 2, 3\}$. Let v be a vertex of G not in H. Then, $N_G(v) \cap H$ is nonempty because net $\cup K_1$ is not a CA graph. If $|N_G(v) \cap H| \geq 5$, then G would contain an induced gem, so $|N_G(v) \cap H| \leq 4$.

Suppose that $|N_G(v) \cap H| = 4$. If $|N_G(v) \cap S| = 3$ then G would contain the graph G_3 in Figure 4 as induced subgraph, which is not a CA graph. If $|N_G(v) \cap S| = 2$, then G would contain an induced gem. So, if $|N_G(v) \cap H| = 4$, then $|N_G(v) \cap S| = 1$.

Suppose, by way of contradiction, that $|N_G(v) \cap H| = 3$. If $|N_G(v) \cap S| = 3$, then G would contain the graph G_9 in Figure 4 as induced subgraph, which is not a CA graph. If $|N_G(v) \cap S| = 2$, then G would contain either $C_5 \cup K_1$ or $C_4 \cup K_1$ as induced subgraph, and neither of them is a CA graph. If $|N_G(v) \cap S| = 1$, then G would contain either a gem or $C_4 \cup K_1$ as induced subgraph. If $|N_G(v) \cap S| = 0$, then G would contain the graph G_6 in Figure 4 as induced subgraph, which is not a CA graph. We conclude that $|N_G(v) \cap H| \neq 3$.

Suppose now that $|N_G(v) \cap H| = 2$. If $|N_G(v) \cap S| = 2$, then G would contain $C_5 \cup K_1$ as induced subgraph, which is not a CA graph. If $|N_G(v) \cap S| = 1$ and the neighbors of v in H were nonadjacent, then G would contain $C_4 \cup K_1$ as induced subgraph. So, if $|N_G(v) \cap H| = 2$, then either $N_G(v) \cap H \subseteq T$ or $N_G(v) \cap H = \{t_i, s_i\}$ for some $i \in \{1, 2, 3\}$.

Finally, if $|N_G(v) \cap H| = 1$, then the neighbor of v in H belongs to T; since otherwise G would contain the graph G_5 in Figure 4 as induced subgraph, and it is not a CA graph.

This shows that we can classify the vertices in $G-H$ as follows: Let S_i be the set of vertices in $G-H$ whose only neighbor in T is t_i (i.e., the set of neighbors in H is either $\{t_i\}$ or $\{t_i, s_i\}$), T_i be the set of vertices in $G-H$ whose neighbors in H are $\{t_1, t_2, t_3, s_i\}$, and Z_i be the set of vertices in $G-H$ whose neighbors in H are $T-\{t_i\}$. Since G is gem-free, at most one of the Z_i's is nonempty. So, without loss of generality, assume that Z_2 and Z_3 are empty.

Claim 8 *S_i is anticomplete to S_j for every $i \neq j$.*

Indeed, if $v \in S_i$ were adjacent to $w \in S_j$ and $i \neq j$, then $\{v, t_i, t_j, w, s_{6-i-j}\}$ would induce a $C_4 \cup K_1$ in G, which is not a CA graph, a contradiction.

Claim 9 *For each $i \in \{1,2,3\}$, S_i is complete to T_i and anticomplete to T_j for every $j \neq i$.*

If $v \in S_i$ and $w \in T_i$ were nonadjacent, then $(H \setminus \{s_1\}) \cup \{v, w\}$ would induce the graph G_6 in Figure 4, which is not a CA graph, a contradiction. If $v \in S_i$ were adjacent to $w \in T_j$ and $j \neq i$, then $\{s_j, t_j, t_i, v, w\}$ would induce a gem in G, a contradiction.

Claim 10 *For each $i \in \{1,2,3\}$, T_i is a complete and T_i is complete to T_j for every $j \neq i$.*

Indeed, if $w, w' \in T_i$ were nonadjacent, then $\{w, s_i, w', t_{i+1}, s_{i+2}\}$ would induce $C_4 \cup K_1$ in G, which is not a CA graph, a contradiction. Also, if $w_i \in T_i$ were nonadjacent to $w_j \in T_j$ and $j \neq i$, then $\{s_j, w_j, t_i, w_i, t_j\}$ would induce a gem in G, a contradiction.

Claim 11 *For each $i \in \{1,2,3\}$, S_i is anticomplete to Z_1.*

Indeed, if $v \in S_i$ were adjacent to $z_1 \in Z_1$, then either $i = 1$ and $\{v, t_1, t_2, z_1, s_3\}$ would induce $C_4 \cup K_1$ in G, or $i \neq 1$ and $\{t_1, t_{5-i}, z_1, v, t_i\}$ would induce gem in G, and in both cases we would reach a contradiction.

Claim 12 *T_1 is anticomplete to Z_1.*

Indeed, if $w_1 \in T_1$ were adjacent to $z_1 \in Z_1$, then $\{s_1, t_1, t_2, z_1, w_1\}$ would induce a gem in G, a contradiction.

By the previous claims, every vertex in T_1 is a true twin of t_1 and, since there are no true twins in G, T_1 is empty. Since the claims also prove that $S_1 \cup \{s_1\}$ is anticomplete to $V(G-\{t_1\}) \setminus (S_1 \cup \{s_1\})$, t_1 is a cutpoint of G, as required. □

The following is an immediate consequence of Theorem 1.

Corollary 13 *Minimally non-balanced graphs have no cutpoints.*

Proof: Let H be a minimally non-balanced graph. By Theorem 1, H is an extended odd sun. Let C and $\{W_e\}_{e \in E(C)}$ be as in the definition of extended odd suns. It is clear that neither the vertices of C nor the vertices of the W_e's are cutpoints. Since $V(H) = V(C) \cup \bigcup_{e \in E(C)} W_e$, H has no cutpoints. □

Now we are ready to characterize balanced graphs among gem-free CA graphs.

Theorem 14 *Let G be a gem-free CA graph. Then, G is balanced if and only if G has no odd holes and contains no induced 3-pyramid.*

Tab. 1: Minimal forbidden induced subgraphs for balancedness in each studied subclass of CA graphs

Subclass of CA graphs	Minimal forbidden induced subgraphs for balancedness	Reference
$\{$net,U_4,$S_4\}$-free CA graphs (contains all HCA graphs)	odd holes, pyramids, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, and X_p^{2t+1}	Corollary 8
claw-free CA graphs (contains all PCA graphs)	odd holes, pyramids, and $\overline{C_7}$	Theorem 11
gem-free CA graphs	odd holes and 3-pyramid	Theorem 14

Proof: The 'only if' part is clear. In order to prove the 'if' part, suppose that G is not balanced. Then, G contains some induced subgraph H that is minimally non-balanced. Clearly, H is a gem-free CA-graph because G is so. The proof will be complete as soon as we prove that H is an odd hole or a 3-pyramid. Suppose, by way of contradiction, that H is not $\{$net,U_4,$S_4\}$-free. Since H is gem-free, H contains an induced net or an induced U_4. By Lemma 12, H has true twins or has a cutpoint, a contradiction with the minimality of H. This contradiction proves that H is $\{$net,U_4,$S_4\}$-free and Corollary 8 implies that H has an odd hole or contains an induced 3-pyramid (because each of 3-sun, 1-pyramid, 2-pyramid, $\overline{C_7}$, X_p^{2t+1}, D^{2t+1}, and X_p^{2t+1}, for each $t \geq 2$ and each valid p, contains an induced gem). The minimality of H ensures that H is an odd hole or a 3-pyramid, which concludes the proof. □

We conclude this section with Table 1 that summarizes the characterizations of balanced graphs by minimal forbidden induced subgraphs within each studied subclass of CA graphs.

3 Considering further superclasses of balanced graphs

Recall that well-known superclasses of balanced graphs are perfect graphs and hereditary clique-Helly graphs. As perfect graphs are exactly the odd hole- and odd antihole-free graphs [14] and hereditary clique-Helly graphs exactly the pyramid-free graphs [28], the list of minimal forbidden induced subgraphs from Theorems 11 and 14 implies the following for claw-free and gem-free CA graphs:

Corollary 15 *Let G be a claw-free CA graph. Then, G is balanced if and only if G is perfect and hereditary clique-Helly.*

Corollary 16 *Let G be a gem-free CA graph. Then, G is balanced if and only if G is perfect and hereditary clique-Helly.*

This motivates us to also consider the relationship between CA graphs and further superclasses of balanced graphs. Note that perfect graphs are characterized in several different ways, among them in terms of a min-max relation of two graph parameters: The size of a largest clique in a graph is a trivial lower bound for the minimum number of colors required to assign different colors to adjacent vertices; perfect graphs are exactly the graphs where both parameters coincide for all induced subgraphs. Some other graph classes were defined in a similar way and turned out to be superclasses of balanced graphs.

A clique-independent set of a graph G is a set of pairwise disjoint cliques of G. A clique-transversal of G is a subset of vertices meeting all the cliques of G. The maximum cardinality $\alpha_c(G)$ of a clique-independent set is a trivial lower bound for the minimum cardinality $\tau_{\mathrm{c}}(G)$ of a clique-transversal of G. A graph G is *clique-perfect* if and only if $\alpha_{\mathrm{c}}(H) = \tau_{\mathrm{c}}(H)$ holds for each induced subgraph H of G. The term 'clique-perfect' was coined in [20], but the equality of these parameters has been studied long before in the context of hypergraphs [4]. It is important to mention that clique-perfect graphs do not need to be perfect; e.g., odd antiholes of length $6n + 3$ are clique-perfect for each $n \geq 1$ and are not perfect (Reed, 2001, cf. [16]).

Coordinated graphs were introduced while looking for characterizations of clique-perfect graphs and are also defined by requiring equality in a min-max inequality between two graph parameters. Let $\gamma_{\mathrm{c}}(G)$ be the minimum number of colors needed to assign different colors to intersecting cliques of G, and let $\Delta_{\mathrm{c}}(G)$ be the maximum cardinality of a family of cliques having a vertex of G in common. Clearly, $\gamma_{\mathrm{c}}(G) \geq \Delta_{\mathrm{c}}(G)$ holds for any graph G, and a graph G is called *coordinated* if $\gamma_{\mathrm{c}}(H) = \Delta_{\mathrm{c}}(H)$ for each induced subgraph H of G. Coordinated graphs form a proper subclass of perfect graphs [8]. Furthermore, the works of Berge [2] and Lovász [26] on hypergraphs imply that a balanced graph is simultaneously hereditary clique-Helly, clique-perfect, and coordinated.

The characterization by forbidden induced subgraphs of clique-perfect and coordinated graphs is open, but some partial results were obtained in [5, 6, 12, 13]. In particular, in [6], clique-perfect graphs were characterized by minimal forbidden induced subgraphs within the class of Helly CA graphs. We present here the minimal forbidden induced subgraph characterization of clique-perfectness and coordination restricted to gem-free CA graphs.

Theorem 17 *Let G be a gem-free CA graph. Then, the following statements are equivalent:*

(i) *G is clique-perfect.*

(ii) *G is coordinated.*

(iii) *G is perfect.*

(iv) *G has no odd holes.*

Proof: A graph is *minimally not clique-perfect* (resp. *minimally not coordinated*) if it is not clique-perfect (resp. not coordinated) but all its proper induced subgraphs are so. Along this proof, we will denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the classes of minimally not clique-perfect and minimally not coordinated graphs, respectively, within the class of gem-free CA graphs. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Clearly, odd holes are in $\mathcal{C}_1 \cap \mathcal{C}_2$ and therefore also in $\mathcal{C}$. If we prove that the odd holes are the only graphs in $\mathcal{C}$, then the equivalence among (i), (ii), and (iv) follows. The equivalence between (iii) and (iv) is an immediate consequence of the Strong Perfect Graph Theorem [14] because every antihole $\overline{C_{2t+1}}$, for $t \geq 3$, contains an induced gem.

Suppose, by way of contradiction, that there exists a graph H in $\mathcal{C}$ that is not an odd hole. In particular, H is not balanced and, by Theorem 14, H contains an induced 3-pyramid. Let $P \subseteq V(H)$ such that P induces a 3-pyramid in H and let $W \subseteq P$ such that W induces a C_4 in H. Let $w_1w_2w_3w_4w_1$ be the hole induced by the vertices of W in H and let $P \setminus W = \{u_1, u_2\}$. Let U be the set of vertices of $V(H) \setminus W$ that are complete to W and, for each $i \in \{1, 2, 3, 4\}$, let V_i be the set of vertices of $V(H) \setminus W$ whose only non-neighbor in W is w_i. Clearly, u_1 and u_2 belong to U.

Claim 1 *Each vertex of H belongs to exactly one of the sets U, V_1, V_2, V_3, V_4, and W.*

The sets U, V_1, V_2, V_3, V_4, and W are pairwise disjoint by definition. Let v be an arbitrary vertex of $V(H) \setminus W$. If $v \in P \setminus W$, then $v \in U$ by construction. So, without loss of generality, suppose that $v \in V(H) \setminus P$. Let $k = |N_H(v) \cap W|$. By Lemma 2 and symmetry, we can assume, without loss of generality, that $N_H(v) \cap W = \{w_i : 1 \leq i \leq k\}$. If $k = 0$, then $V(H) \cup \{v\}$ would induce $C_4 \cup K_1$ in H, which is not a CA graph, a contradiction. If $k = 1$, then either v would be adjacent to u_i for some $i \in \{1, 2\}$ and $\{v, w_1, w_2, w_3, u_i\}$ would induce a gem in H or $\{u_1, w_2, u_2, w_4, v\}$ would induce $C_4 \cup K_1$ in H, a contradiction. If $k = 2$, either $\{v, w_1, w_4, w_3, u_1\}$ or $\{v, w_2, u_1, w_4, w_1\}$ induces a gem in H depending on whether v is adjacent to u_1 or not, respectively, another contradiction. We conclude that either $k = 3$ or $k = 4$; i.e., $v \in V_i$ for some $i \in \{1, 2, 3, 4\}$ or $v \in U$, as claimed.

Claim 2 *For each $i \in \{1, 2, 3, 4\}$, every vertex of V_i is a true twin of w_i in H.*

In fact, notice that the following statements hold for each $i \in \{1, 2, 3, 4\}$ (subindices should be understood modulo 4):

- *V_i is complete to U*: If there were vertices $v_i \in V_i$ and $u \in U$ which were nonadjacent, then $\{u, v_i, w_i, w_{i+1}, w_{i+3}\}$ would induce a $K_{2,3}$ in H, a contradiction.
- *V_i is a complete set*: If there were two different nonadjacent vertices v_i and v'_i in V_i, then the set $\{v_i, v'_i, w_i, w_{i+1}, w_{i+3}\}$ would induce a $K_{2,3}$ in H, a contradiction.
- *V_i is complete to V_{i+1}*: If there were vertices v_i in V_i and v_{i+1} in V_{i+1} which were nonadjacent, then $\{v_i, w_{i+2}, v_{i+1}, w_i, w_{i+3}\}$ would induce a gem in H, a contradiction.
- *V_i is anticomplete to V_{i+2}*: If there were vertices v_i in V_i and v_{i+2} in V_{i+2} which were adjacent, then $\{w_i, v_{i+2}, v_i, w_{i+2}, w_{i+1}\}$ would induce a gem in H, a contradiction.

By Claim 1, the definition of V_i, and the four above statements, it follows that every vertex of V_i is a true twin of w_i in H, as claimed.

Claim 3 *The sets V_1, V_2, V_3, and V_4 are all empty.*

Indeed, since α_c, τ_c, γ_c, and Δ_c are invariant under the addition of a true twin, no graph in $\mathcal{C}_1 \cup \mathcal{C}_2$ has true twins. In particular, H has no true twins and, by Claim 2, V_i is empty for each $i \in \{1, 2, 3, 4\}$, as claimed.

Claim 4 *$\overline{H}$ is the disjoint union of three or more complete bipartite graphs. (We regard an isolated vertex as a trivial complete bipartite graph.)*

In fact, since H is gem-free and $K_{2,3}$-free, $H[U]$ is P_4-free and $3K_1$-free. Moreover, in [7], it is proved that if a graph is $\{3K_1, P_4\}$-free, then its complement is the disjoint union of complete bipartite graphs. Therefore, $\overline{H}[U]$ is the disjoint union of complete bipartite graphs. By Claims 1 and 3, $V(H) = U \cup W$. Since $\overline{H}[W] = \overline{C_4} = 2K_2$ and U is anticomplete to W in $\overline{H}$, $\overline{H}$ is the disjoint union of at least three complete bipartite graphs, as claimed.

With the help of Claim 4 we now complete the proof of the theorem. Notice that Claim 4 implies that $H \notin \mathcal{C}_1$ because no minimally not clique-perfect graph is the complement of a disconnected graph (cf. [25, Lemma 1]). So, necessarily, $H \in \mathcal{C}_2$; i.e., H is minimally not coordinated. Then, $\gamma_c(H) \neq \Delta_c(H)$ and, in particular, H has no universal vertices; i.e., each connected component of $\overline{H}$ has at least two vertices.

Let $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_t$ be the connected components of $\overline{H}$ and, for each $i \in \{1, 2, \ldots, t\}$, let $\{A_i^1, A_i^2\}$ be the partition into two stable sets of the vertices of the complete bipartite graph $\overline{H}_i$. Then, the cliques of H are of the form $A_1^{j_1} \cup A_2^{j_2} \cup \cdots \cup A_t^{j_t}$ where $j_1, \ldots, j_t \in \{1, 2\}$. Notice that $\gamma_c(H) = 2^{t-1}$ and clearly $\Delta_c(H) = 2^{t-1}$ (indeed, each vertex of H belongs to 2^{t-1} cliques of H), which contradicts the fact that $H \in \mathcal{C}_2$. This contradiction arose from assuming that there was some graph in $\mathcal{C}_1 \cup \mathcal{C}_2$ which was not an odd hole, and thus completes the proof of the theorem. □

4 Further remarks

In this work, we considered the problem of characterizing balanced graphs by minimal forbidden subgraphs within different subclasses of CA graphs, including a superclass of each of two of the most studied subclasses of CA graphs: the class of HCA graphs and the class of PCA graphs. The complete characterization of balanced graphs by minimal forbidden induced subgraphs within CA graphs in general, remains unknown. The sun S_5 is the only example of a minimally non-balanced CA graph not belonging to any of the here studied classes of CA graphs that we know. (The graphs S_t with t odd and $t \geq 7$ are not circular-arc graphs.)

A careful reading of the proof of Theorem 3 reveals that the hypothesis that the graph is HCA (and not merely a CA graph) is only used in the proofs of Claim 1 and 2, and in the latter case only for $t = 2$. So, along the proof we indeed identified all CA graphs that are minimally non-balanced and whose unbalanced cycles have length at least 7 and have only short chords. Therefore, a possible road towards extending the proof of Theorem 3 to the entire class of CA graphs could be that of finding some property of the chords of the unbalanced cycles within CA graphs in general that could serve as a substitute for Claim 1. A different approach would be to take Theorem 6 as a starting point and to study the balancedness of CA graphs containing net, U_4, or S_4 as induced subgraph. We managed to do so when restricting ourselves to claw-free and gem-free graphs. A better understanding of the structure of CA graphs would be of help to overcome these restrictions.

We also considered the problem of characterizing clique-perfect and coordinated graphs by forbidden induced subgraphs within CA graphs. In [6], clique-perfect graphs were characterized by minimal forbidden induced subgraphs within the class of HCA graphs. It is easy to see that the approach used here to extend Theorem 3 to Corollary 8, works also for extending the characterization of clique-perfect graphs within HCA graphs in [6] to a characterization of clique-perfect graphs within the class of $\{$net,U_4,$S_4\}$-free $\cap$ $\{$1-pyramid,2-pyramid,3-pyramid$\}$-free CA graphs. Nevertheless, characterizing clique-perfect graphs by forbidden induced subgraphs within all $\{$net,U_4,$S_4\}$-free CA graphs seems much harder. The characterization of clique-perfect graphs by forbidden induced subgraphs is open even within PCA graphs. For coordinated graphs, the characterization remains unresolved even within both HCA graphs and PCA graphs.

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