On the bend number of circular-arc graphs as edge intersection graphs of paths on a grid

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Abstract

Golumbic, Lipshteyn and Stern proved that every graph can be represented as the edge intersection graph of paths on a grid, i.e., one can associate to each vertex of

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This work was partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France), UBACyT Grant 20020130100808BA, CONICET PIP 122-01001-00310, 112-200901-00178 and 112-201201-00450CO and ANPCyT PICT 2010-1970 and 2012-1324 (Argentina), FONDECyT Grant 1140787 and Millennium Science Institute "Complex Engineering Systems" (Chile).

the graph a nontrivial path on a grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. For a nonnegative integer k, B_k-EPG graphs are defined as graphs admitting a model in which each path has at most k bends. Circular-arc graphs are intersection graphs of open arcs of a circle. It is easy to see that every circular-arc graph is B_4-EPG, by embedding the circle into a rectangle of the grid. In this paper we prove that every circular-arc graph is B_3-EPG, but if we restrict ourselves to rectangular representations there exist some graphs that require paths with four bends. We also show that normal circular-arc graphs admit rectangular representations with at most two bends per path. Moreover, we characterize graphs admitting a rectangular representation with at most one bend per path by forbidden induced subgraphs, and we show that they are a subclass of normal Helly circular-arc graphs.

Keywords: edge intersection graphs, paths on a grid, forbidden induced subgraphs, (normal, Helly) circular-arc graphs.

1 Introduction

Edge intersection graphs of paths on a grid (EPG graphs) are graphs whose vertices can be represented as nontrivial paths on a grid such that two vertices are adjacent if and only if the corresponding paths share at least one edge of the grid. Every graph can be represented in such a way on a large enough grid and allowing an arbitrary number of *bends* (turns on a grid point) for each path [10]. In recent years, the subclasses in which the number of bends of each path is bounded by some number k, known as B_k-EPG graphs, were widely studied [2,1,3,7,10]. For instance, it is easy to see that B_0-EPG graphs are exactly the class of interval graphs (intersection graphs of intervals on a line) [10]. Similarly, a natural representation of circular-arc graphs (intersection graphs of open arcs on a circle) as EPG graphs arises by identifying the circle with a rectangle of the grid. So circular-arc graphs are a subclass of B_4-EPG graphs. This leads to some natural questions, as for example the existence of $k < 4$ such that circular-arc graphs are a subclass of B_k-EPG graphs, and the characterization of circular-arc graphs that are B_k-EPG graphs, for $k < 4$. Also, how many bends per path are needed for a circular-arc graph to be represented in a rectangle, i.e., in such a way that the union of the paths is contained in a rectangle of the grid.

One of the main results of this paper is to prove that circular-arc graphs are a subclass of B_3-EPG graphs, and that there exist some circular-arc graphs in B_3-EPG $\setminus$ B_2-EPG. We also consider here EPG representations in which the union of the paths is contained in a rectangle of the grid. We will call these graphs *edge intersection graphs of paths on a rectangle* (or for short EPR graphs). It is easy to see that EPR graphs are exactly the circular-arc graphs. We will study the classes B_k-EPR, for $0 \leq k \leq 4$, in which the paths on the grid that represent the vertices of G have at most k bends.

In this paper we focus on B_1-EPR graphs and B_2-EPR graphs (B_0-EPR graphs are the class of interval graphs), and relate these classes with the class of normal Helly circular-arc graphs. In summary, the contributions of this paper are: we prove that normal circular-arc graphs are B_2-EPR; moreover, we show that B_1-EPR graphs are normal Helly circular-arc; finally, we characterize by forbidden induced subgraphs B_1-EPR graphs, and show that they are a subclass of NHCA graphs. We will also show, for completeness, that there are graphs in B_4-EPR $\setminus$ B_3-EPR, and in B_3-EPR $\setminus$ B_2-EPR.

2 Preliminaries

In this paper all graphs are connected, finite and simple. Notation we use is that used by Bondy and Murty [4].

We will denote by C_n the chordless cycle of n vertices and by $\overline{C_n}$ its complement. A *thick spider* S_n is the graph whose $2n$ vertices can be partitioned into a complete $c_1, \ldots, c_n$ and a stable set $s_1, \ldots, s_n$ in such a way that, for $1 \leq i, j, \leq n$, c_i is adjacent to s_j if and only if $i \neq j$. Note that S_k is an induced subgraph of S_n if $k \leq n$.

A graph G is a *circular-arc graph* (for short CA graph) if it is the vertex intersection graph of a set $\mathcal{A}$ of open arcs on a circle $\mathcal{C}$, and $(\mathcal{A}, \mathcal{C})$ is called a *circular-arc model* of G [13]. A graph G is a *Helly circular-arc graph* (or for short HCA graph) if it is a circular-arc graph having a circular-arc model such that any subset of pairwise intersecting arcs has a common point on the circle [9]. A circular-arc graph having a circular-arc model without two arcs covering the whole circle is called a *normal circular-arc graph* (or for short NCA graph). Circular-arc models that are at the same time normal and Helly are precisely those without three or less arcs covering the whole circle. A graph that admits such a model is called a *normal Helly circular-arc graph* (or for short NHCA graph) [11].

In [5], Cao, Grippo and Safe give a characterization of NHCA graphs by forbidden induced subgraphs. Recent surveys on circular-arc graphs are [6,12]. A very recent characterization of circular-arc graphs by forbidden structures can be found in [8].

3 Main results

One of our main results is the following.

Theorem 3.1 *Every circular-arc graph is B_3-EPG. The thick spider S_{46} is in B_3-EPG $\setminus$ B_2-EPG.*

Proof. Let G be a circular-arc graph and let $(\mathcal{A}, \mathcal{C})$ be a circular-arc model of G. W.l.o.g., we may assume that the endpoints of the arcs are all distinct and we can number them clockwise in the circle from 1 to $2n$ (being n the number of vertices of G) and define a point 0 in the circle between $2n$ and 1 (clockwise). The arc (a, b), $1 \leq a, b \leq 2n$, means the arc in the circle traversing clockwise from point a to point b. In particular, and arc (a, b) contains point 0 of $\mathcal{C}$ if and only if $a > b$. The set of vertices X of G corresponding to the arcs containing point 0 of $\mathcal{C}$ induce a complete subgraph on G. Moreover, $G - X$ is an interval graph that can be represented on a line by taking, for each vertex, the interval (a, b) defined by the endpoints of its corresponding arc, since $a < b$ for vertices of $G - X$. We will construct the following model of G on a grid. For each vertex of $G - X$ corresponding to an arc (a, b), assign the 3-bends-path on the grid whose vertices are $(0, b), (0, a), (a, a), (a, 0), (b, 0)$. For each vertex of X corresponding to an arc (a, b) (in this case $a > b$), assign the 3-bends-path on the grid whose vertices are $(0, 0), (0, b), (a, b), (a, 0), (n+1, 0)$. Since all the endpoints of the arcs in $\mathcal{A}$ are different, the edge intersections of the paths are either on row 0 or on column 0 of the grid. Two intervals corresponding to vertices of $G - X$ intersect if and only if the corresponding arcs intersect on $\mathcal{C}$. Two intervals corresponding to vertices of X intersect at least at the edge of the grid $(0, 0), (0, 1)$. The interval corresponding to a vertex in $G - X$ with endpoints (a, b) and the interval corresponding to a vertex in X with endpoints (c, d) intersect if and only if either $d > a$ or $c < b$, and the same condition holds for the arcs in $\mathcal{C}$. The proof of S_{46} being in B_3-EPG $\setminus$ B_2-EPG is omitted due to lack of space. □

We do not know if 46 is the minimum k such that $S_k \in B_3$-EPG $\setminus$ B_2-EPG, but for S_{46} the proof is very simple. We leave as an open problem the characterization of AC $\cap$ B_2-EPG and AC $\cap$ B_1-EPG by minimal forbidden induced subgraphs.

In the following, we focus on circular-arc graphs representations as edge intersection graphs of paths with a bounded number of bends on a rectangle of the grid.

Theorem 3.2 *NCA $\subsetneq$ B_2-EPR.*

Proof. Let $(\mathcal{A}, \mathcal{C})$ be a NCA model of a graph. W.l.o.g., we may assume that the endpoints of the arcs are pairwise different. Let p be a point of $\mathcal{C}$ that is not the endpoint of an arc of $\mathcal{A}$. Since the model is normal, the union of the arcs of $\mathcal{A}$ that contain p does not cover $\mathcal{C}$ so, by our assumption, there is a point q in $\mathcal{C}$ that is not the endpoint of an arc of $\mathcal{A}$ and is not contained in the union of the arcs of $\mathcal{A}$ that contain p. We can then embed our model on a rectangle of the grid in such a way that two consecutive corners correspond to point p of the circle and the remaining two corners correspond to point q of the circle. In this way, since no arc of $\mathcal{A}$ contains both p and q, paths corresponding to arcs containing either p or q have two bends, while paths corresponding to arcs containing neither p nor q have no bends. It can be seen that the thick spider S_6 is in B_2-EPR $\setminus$ NCA. Hence, the inclusion is proper. $\square$

Lemma 3.3 *B_1-EPR $\subsetneq$ NHCA.*

Proof. Let $\langle \mathcal{P}, \mathcal{R} \rangle$ be a B_1-EPR representation of a graph G, where $\mathcal{P}$ is the family of paths with at most one bend each and $\mathcal{R}$ is a rectangle of a grid containing the union of the paths in $\mathcal{P}$. We will define a natural bijection between $\mathcal{R}$ and a circle $\mathcal{C}$, that also maps the paths in $\mathcal{P}$ to open arcs $\mathcal{A}$ of $\mathcal{C}$. Notice that two open arcs intersect if and only if the corresponding paths of $\mathcal{P}$ intersect on an least one edge of the grid. So $(\mathcal{A}, \mathcal{C})$ is a circular-arc representation of G. Now, since each path has at most one bend and the arcs are open, the union of three (resp. two) arcs of $\mathcal{A}$ contains at most three (resp. two) points of $\mathcal{C}$ corresponding to corners of $\mathcal{R}$. In particular, since $\mathcal{R}$ has four corners, it does not cover the whole circle. Hence $(\mathcal{A}, \mathcal{C})$ is a NHCA model for G. It can be seen that the NHCA graph $\overline{C_7}$ is not a B_1-EPG graph, thus is not a B_1-EPR graph. Hence, the inclusion is proper. $\square$

We will prove the following theorem by characterizing the structure of B_1-EPR graphs and their NHCA models.

A *snail* is a claw-free NHCA graph containing an induced C_4, namely $v_1v_2v_3v_4$, and such that initializing $V_i = \{v_i\}$ for $i = 1, \ldots, 4$ and performing the iterative process $V_i = V_i \cup \{v\}$ if v has neighbors in V_{i-1}, V_i and V_{i+1}, at some step there is a vertex having neighbors in every V_i, for $i = 1, \ldots, 4$. For example, the graph $\overline{C_7}$ is a snail.

Theorem 3.4 *$G \in B_1$-EPR if and only if $G \in$ NHCA and G has no snail as induced subgraph.*

Proposition 3.5 *The thick spiders S_3, S_7, and S_{13}, belong to B_2-EPR $\setminus$ B_1-EPR, B_3-EPR $\setminus$ B_2-EPR, and B_4-EPR $\setminus$ B_3-EPR, respectively.*

We leave as an open problem the characterization of AC ∩ B_2-EPR and AC ∩ B_3-EPR by minimal forbidden induced subgraphs, and the explicit description of all the minimal snails.

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