k-tuple chromatic number of the cartesian product of graphs [1]

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Abstract

A k-tuple coloring of a graph G assigns a set of k colors to each vertex of G such that if two vertices are adjacent, the corresponding sets of colors are disjoint. The k-tuple chromatic number of G, $\chi_k(G)$, is the smallest t so that there is a k-tuple coloring of G using t colors. It is well known that $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$. In this paper, we show that there exist graphs G and H such that $\chi_k(G \square H) > \max\{\chi_k(G), \chi_k(H)\}$ for $k \geq 2$. Moreover, we also show that there exist graph families such that, for any $k \geq 1$, the k-tuple chromatic number of their cartesian product is equal to the maximum k-tuple chromatic number of its factors.

Keywords: k-tuple colorings, Cartesian product of graphs, Kneser graphs, Cayley graphs, Hom-idempotent graphs.

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1 Introduction

A classic *coloring* of a graph G is an assignment of colors (or natural numbers) to the vertices of G such that any two adjacent vertices are assigned different colors. The smallest number t such that G admits a coloring with t colors (a t-*coloring*) is called the *chromatic number* of G and is denoted by $\chi(G)$. Several generalizations of the coloring problem have been introduced in the literature, in particular, cases in which each vertex is assigned not only a color but a set of colors, under different restrictions. One of these variations is the k-*tuple coloring* introduced independently by Stahl [6] and Bollobás and Thomason [1]. A k-tuple coloring of a graph G is an assignment of k colors to each vertex in such a way that adjacent vertices are assigned distinct colors. The k-*tuple coloring problem* consists into finding the minimum number of colors in a k-tuple coloring of a graph G, which we denote by $\chi_k(G)$. The *cartesian product* $G\square H$ of two graphs G and H has vertex set $V(G) \times V(H)$, two vertices being joined by an edge whenever they have one coordinate equal and the other adjacent. This product is commutative and associative up to isomorphism. There is a simple formula expressing the chromatic number of a cartesian product in terms of its factors:

$$\chi(G\square H) = \max\{\chi(G), \chi(H)\}. \qquad (1)$$

The identity (1) admits a simple proof first given by Sabidussi [5]. The *Kneser graph* $K(m,n)$ has as vertices all n-element subsets of the set $[m] = \{1,\ldots,m\}$ and an edge between two subsets if and only if they are disjoint. We will assume in the rest of this work that $m \geq 2n$, otherwise $K(m,n)$ has no edges. The Kneser graph $K(5,2)$ is the well known *Petersen Graph*. Lovász [4] showed that $\chi(K(m,n)) = m-2n+2$. The value of the k-tuple chromatic number of the Kneser graph is the subject of an almost 40-year-old conjecture of Stahl [6] which asserts that: if $k = qn-r$ where $q \geq 0$ and $0 \leq r < n$, then $\chi_k(K(m,n)) = qm-2r$. Stahl's conjecture has been confirmed for some values of k, n and m [6,7]. An *homomorphism* from a graph G into a graph H, denoted by $G \to H$, is an edge-preserving map from $V(G)$ to $V(H)$. It is well known that an ordinary graph coloring of a graph G with m colors is an homomorphism from G into the complete graph K_m. Similarly, an n-tuple coloring of a graph G with m colors is an homomorphism from G into the Kneser graph $K(m,n)$. A graph G is said *hom-idempotent* if there is a homomorphism from $G\square G \to G$. We denote by $G \not\to H$ if there exists no homomorphism from G to H. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a *clique* in G (i.e., a complete subgraph of

G). Clearly, for any graphs G and H, we have that $\chi(G) \geq \omega(G)$ (and so, $\chi_k(G) \geq \chi_k(K_{\omega(G)}) = k\omega(G)$) and, if there is an homomorphism from G to H then, $\chi(G) \leq \chi(H)$ (and so, $\chi_k(G) \leq \chi_k(H)$).

In this paper, we show that equality (1) does not hold in general for k-tuple colorings of graphs. In fact, we show that for some values of $k \geq 2$, there are Kneser graphs $K(m,n)$ for which $\chi_k(K(m,n)\square K(m,n)) > \chi_k(K(m,n))$. Moreover, we also show that there are families of graphs for which equality (1) holds for k-tuple colorings of graphs for any $k \geq 1$. As far as we know, our results are the first ones concerning the k-tuple chromatic number of cartesian product of graphs.

2 Cartesian products of Kneser graphs

Lemma 2.1 *Let G be a graph and let $k > 0$. Then, $\chi_k(G\square G) \leq k\chi(G)$.*

Proof. Clearly, $\chi_k(G\square G) \leq k\chi(G\square G)$. However, by equality (1) we know that $\chi(G\square G) = \chi(G)$, and thus the lemma holds. □

Corollary 2.2 $\chi_k(K(m,n)\square K(m,n)) \leq k\chi(K(m,n)) = k(m-2n+2)$.

Larose et al. [3] showed that no connected Kneser graph $K(m,n)$ is hom-idempotent, that is, for any $m > 2n$, there is no homomorphism from $K(m,n)\square K(m,n)$ to $K(m,n)$.

Lemma 2.3 **([3])** *Let $m > 2n$. Then, $K(m,n)\square K(m,n) \not\to K(m,n)$.*

Concerning the k-tuple chromatic number of some Kneser graphs, Stahl [6] showed the following results.

Lemma 2.4 **([6])** *If $1 \leq k \leq n$, then $\chi_k(K(m,n)) = m - 2(n-k)$.*

Lemma 2.5 **([6])** $\chi_k(K(2n+1,n)) = 2k+1+\lfloor \frac{k-1}{n} \rfloor$, *for $k > 0$.*

Lemma 2.6 **([6])** $\chi_{rn}(K(m,n)) = rm$, *for $r > 0$ and $m \geq 2n$.*

By using Lemma 2.6 we have the following result.

Lemma 2.7 *Let $m > 2n$. Then, $\chi_n(K(m,n)\square K(m,n)) > \chi_n(K(m,n))$.*

Proof. By Lemma 2.6 when $r = 1$, we have that $\chi_n(K(m,n)) = m$. If $\chi_n(K(m,n)\square K(m,n)) = m$, then there exists an homomorphism from the graph $K(m,n)\square K(m,n)$ to $K(m,n)$ which contradicts Lemma 2.3. □

By Lemma 2.4, Lemma 2.7 and by using Corollary 2.2, we have that,

Corollary 2.8 *Let $n \geq 2$. Then, $2n+2 \leq \chi_n(K(2n+1,n)\Box K(2n+1,n)) \leq 3n$. In particular, when $n=2$, we have that $\chi_2(K(5,2)\Box K(5,2)) = 6$.*

In the case $k=2$ we have by Lemma 2.7, Lemma 2.4 and by Corollary 2.2, the following result.

Corollary 2.9 *Let $q > 0$. Then, $q+4 \leq \chi_2(K(2n+q,n)\Box K(2n+q,n)) \leq 2q+4$.*

By Corollary 2.9, notice that in the case when $k=n=2$ and $q \geq 1$, we must have that $\chi_2(K(q+4,2)\Box K(q+4,2)) > q+4$, otherwise there is a contradiction with Lemma 2.3. This provides a gap of one unity between the 2-tuple chromatic number of the graph $K(q+4,2)\Box K(q+4,2)$ and the graph $K(q+4,2)$. In the next lemma, we show that such a gap can be as large as desired.

Lemma 2.10 *Let $q > 0$. Then, $\chi_2(K(2q+4,2)\Box K(2q+4,2)) \geq 2q + \lceil \frac{2}{3}q \rceil + 5$.*

As a corollary of Lemma 2.10 and by Corollary 2.2, we obtain the following result.

Corollary 2.11 $\chi_2(K(6,2)\Box K(6,2)) = 8$.

Theorem 2.12 *Let $k > n$ and let $t = \chi_k(K(m,n)\Box K(m,n))$, where $m > 2n$. Then, either $t > m + 2(k-n)$ or $t < m + (k-n)$.*

Proof. Suppose that $m + (k-n) \leq t \leq m + 2(k-n)$. Therefore, there exists an homomorphism $K(m,n)\Box K(m,n) \to K(t,k)$. Now, Stahl [6] showed that there is an homomorphism $K(m,n) \to K(m-2,n-1)$ whenever $n > 1$ and $m \geq 2n$. Moreover, it is easy to see that there is an homomorphism $K(m,n) \to K(m-1,n-1)$. By applying the former homomorphism $t-(m+(k-n))$ times to the graph $K(t,k)$ we obtain an homomorphism $K(t,k) \to K(2(m+k-n)-t, 2k+m-n-t)$. Finally, by applying $2k+m-t-2n$ times the latter homomorphism to the graph $K(2(m+k-n)-t, 2k+m-n-t)$ we obtain an homomorphism $K(2(m+k-n)-t, 2k+m-n-t) \to K(m,n)$. Therefore, by homomorphism composition, $K(m,n)\Box K(m,n) \to K(m,n)$ which contradicts Lemma 2.3. $\Box$

We can also obtain a lower bound for the k-tuple chromatic number of the graph $K(m,n)\Box K(m,n)$ in terms of the clique number of $K(m,n)$. In fact, notice that $\omega(K(m,n)\Box K(m,n)) = \omega(K(m,n)) = \lfloor \frac{m}{n} \rfloor$. Thus, we have that $\chi_k(K(m,n)\Box K(m,n)) \geq k\omega(K(m,n)) = k\lfloor \frac{m}{n} \rfloor$.

3 Cases where $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$

Theorem 3.1 *Let G and H be graphs such that $\chi(G) \leq \chi(H) = \omega(H)$. Then, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $t = \omega(H)$ and let $\{h_1, \ldots, h_t\}$ be the vertex set of a maximum clique K_t in H with size t. Clearly, $\chi_k(G) \leq \chi_k(H) = \chi_k(K_t)$. Let ρ be a k-tuple coloring of H with $\chi_k(H)$ colors. By equality (1), there exists a t-coloring f of $G\Box H$. Therefore, the assignment of the k-set $\rho(h_{f((a,b))})$ to each vertex (a, b) in $G\Box H$ defines a k-tuple coloring of $G\Box H$ with $\chi_k(K_t)$ colors.□

Notice that if G and H are both bipartite, then $\chi_k(G\Box H) = \chi_k(G) = \chi_k(H)$. In the case when G is not a bipartite graph, we have the following results.

An *automorphism* σ of a graph G is called a *shift* of G if $\{u, \sigma(u)\} \in E(G)$ for each $u \in V(G)$ [3]. In other words, a shift of G maps every vertex to one of its neighbors.

Theorem 3.2 *Let G be a non bipartite graph having a shift $\sigma \in AUT(G)$, and let H be a bipartite graph. Then, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\}$.*

Proof. Let $A \cup B$ be a bipartition of the vertex set of H. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. Clearly, $\chi_k(G) \geq \chi_k(H)$. We define a k-tuple coloring ρ of $G\Box H$ with $\chi_k(G)$ colors as follows: for any vertex (u, v) of $G\Box H$ with $u \in G$ and $v \in H$, define $\rho((u, v)) = f(u)$ if $v \in A$, and $\rho((u, v)) = f(\sigma(u))$ if $v \in B$. □

We may also deduce the following direct result.

Theorem 3.3 *Let G be an hom-idempotent graph an let H be a subgraph of G. Thus, $\chi_k(G\Box H) = \max\{\chi_k(G), \chi_k(H)\} = \chi_k(G)$.*

Let A be a group and S a subset of A that is closed under inverses and does not contain the identity. The *Cayley graph $Cay(A, S)$* is the graph whose vertex set is A, two vertices u, v being joined by an edge if $u^{-1}v \in S$. If $a^{-1}Sa = S$ for all $a \in A$, then $Cay(A, S)$ is called a *normal Cayley graph.*

Lemma 3.4 ([2]) *Any normal Cayley graph is hom-idempotent.*

Note that all Cayley graphs on abelian groups are normal, and thus hom-idempotents. In particular, the *circulant* graphs are Cayley graphs on cyclic groups (i.e., cycles, powers of cycles, complements of powers of cycles, complete graphs, etc). By Theorem 3.3 and Lemma 3.4 we have the following result.

Theorem 3.5 *Let $Cay(A, S)$ be a normal Cayley graph and let $Cay(A', S')$ be a subgraph of $Cay(A, S)$, with $A' \subseteq A$ and $S' \subseteq S$. Then,* $\chi_k(Cay(A, S)\Box Cay(A', S')) = \max\{\chi_k(Cay(A, S)), \chi_k(Cay(A', S'))\}$.

Definition 3.6 Let G be a graph with a shift σ. We define the *order* of σ as the minimum integer i such that σ^i is equal to the identity permutation.

Theorem 3.7 *Let G be a graph with a shift σ of minimum odd order $2s + 1$ and let C_{2t+1} be a cycle graph, where $t \geq s$. Then,* $\chi_k(G\Box C_{2t+1}) = \max\{\chi_k(G), \chi_k(C_{2t+1})\}$.

Proof. Let $\{0, \ldots, 2t\}$ be the vertex set of C_{2t+1}, where for $0 \leq i \leq 2t$, $\{i, i+1 \mod n\} \in E(C_{2t+1})$. Let G_i be the i^{th} copy of G in $G\Box C_{2t+1}$, that is, for each $0 \leq i \leq 2t$, $G_i = \{(g, i) : g \in G\}$. Let f be a k-tuple coloring of G with $\chi_k(G)$ colors. We define a k-tuple coloring of $G\Box C_{2t+1}$ with $\chi_k(G)$ colors as follows: let σ^0 denotes the identity permutation of the vertices in G. Now, for $0 \leq i \leq 2s$, assign to each vertex $(u, i) \in G_i$ the k-tuple $f(\sigma^i(u))$. For $2s + 1 \leq j \leq 2t$, assign to each vertex $(u, j) \in G_j$ the k-tuple $f(u)$ if j is odd, otherwise, assign to (u, j) the k-tuple $f(\sigma^1(u))$. It's not difficult to see that this is in fact a proper k-tuple coloring of $G\Box C_{2t+1}$. $\Box$

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